ARMA/USRMS 05-810

The Shear Strength Reduction Method for the Generalized Hoek-Brown Criterion

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This paper was prepared for presentation at Alaska Rocks 2005, The 40th U.S. Symposium on Rock Mechanics (USRMS): Rock Mechanics for Energy, Mineral and Infrastructure Development in the Northern Regions, held in Anchorage, Alaska, June 25-29, 2005.

ABSTRACT: This paper describes a method that allows direct use of the Generalized Hoek-Brown criterion in Finite Element (FE) Shear Strength Reduction (SSR) analysis of rock slopes. It describes the generation of shear envelopes for the criterion, lowering of the shear envelope by a factor, and means for determining an equivalent Generalized Hoek-Brown curve that best approximates the lowered envelope. The paper provides two examples that demonstrate the performance of the new method. As well it outlines some of the benefits of the SSR technique for slope stability analysis.

1. INTRODUCTION

The Finite Element Method (FEM) is increasingly being applied to slope stability analysis. One of the most popular techniques for performing FEM slope analysis is the shear strength reduction (SSR) approach [1]. The SSR is simple in concept: systematically reduce the shear strength envelope of material by a factor of safety, and compute FEM models of the slope until deformations are unacceptably large or solutions do not converge.

A factor limiting broader application of the SSR approach to slope stability analysis has been its restriction to Mohr-Coulomb materials. Most discussions of the method found in literature deal with this criterion (the paper by Shukra and Baker [2] is one of the few known to the authors that examines application to non-linear strength envelopes). For rock masses, the Generalized Hoek-Brown criterion is the most commonly applied strength model. As a result, the authors found it expedient to develop an SSR framework for the Hoek-Brown criterion. The aim of this paper is to outline an approach for applying the Generalized Hoek-Brown criterion in SSR analysis. The paper will also demonstrate the capability and accuracy of the proposed approach through two examples.

2. OVERVIEW OF THE SHEAR STRENGTH REDUCTION METHOD

The SSR technique for slope stability analysis involves systematic use of finite element analysis to determine a stress reduction factor (SRF) or factor of safety value that brings a slope to the verge of failure. The shear strengths of all the materials in a FE model of a slope are reduced by the SRF. Conventional FE analysis of this model is then performed until a critical SRF value that induces instability is attained. A slope is considered unstable in the SSR technique when its FE model does not converge to a solution (within a specified tolerance).

As mentioned earlier, most existing descriptions and discussions of the SSR technique are based on use of the Mohr-Coulomb strength models for materials. The criterion is readily used in the SSR technique for the following reasons:

(i) It can be expressed either in terms of principal stresses, or in terms of shear and normal stresses (this makes it amenable for use in both FE and limit-equilibrium analyses)

(ii) Its linearity that allows reduced parameters to be readily calculated when an original shear strength model is reduced by a factor *F* (Griffith and Lane [1] provide simple, closed-form equations for calculating reduced parameters), and

(iii) It is readily available in many existing finite element software.

Determining the parameters of a Generalized Hoek-Brown model, which is equivalent to a shear envelope reduced by a factor of safety, is not as straightforward. This paper will develop an approach for estimating these equivalent parameters.

3. FE SLOPE STABILITY FOR GENERALIZED HOEK-BROWN MATERIALS

The Generalized Hoek-Brown criterion [3] for rock masses is non-linear, and defines material strength in terms of major and minor principal stresses through the equation

$$\sigma_1 = \sigma_3 + \sigma_{ci}\left(m_b \frac{\sigma_3}{\sigma_{ci}} + s\right)^a \quad (1)$$

where σ_{ci} is the uniaxial compressive strength of the intact rock material, while

$m_b = m_i \exp\left(\frac{GSI-100}{28-14D}\right)$, $s = \exp\left(\frac{GSI-100}{9-3D}\right)$, and $a = \frac{1}{2} + \frac{1}{6}\left(e^{-GSI/15} - e^{-20/3}\right)$.

m_i is an intact rock material property, *GSI* is the geological strength index, and *D* is the disturbance factor [3].

Using relationships developed by Balmer [4, 3], a shear-normal stress envelope equivalent to the Generalized Hoek-Brown principal stress envelope can be determined. The shear and normal stress pair corresponding to a point on a principal stress envelope can be determined from the following equations:

$$\sigma_n = \frac{1}{2}(\sigma_1 + \sigma_3) - \frac{1}{2}(\sigma_1 - \sigma_3)\frac{d\sigma_1/d\sigma_3 - 1}{d\sigma_1/d\sigma_3 + 1} \quad (2)$$

$$\tau = (\sigma_1 - \sigma_3)\frac{\sqrt{d\sigma_1/d\sigma_3}}{d\sigma_1/d\sigma_3 + 1}. \quad (3)$$

For the Generalized Hoek-Brown criterion, the following equations relate σ_n and τ to σ_1 and σ_3:

$$\tau = (\sigma_1 - \sigma_3)\frac{\sqrt{1 + am_b\left(m_b \frac{\sigma_3}{\sigma_{ci}} + s\right)^{a-1}}}{2 + am_b\left(m_b \frac{\sigma_3}{\sigma_{ci}} + s\right)^{a-1}} \quad (4)$$

$$\sigma_n = \frac{1}{2}(\sigma_1 + \sigma_3) - \frac{1}{2}(\sigma_1 - \sigma_3)\frac{am_b\left(m_b \frac{\sigma_3}{\sigma_{ci}} + s\right)^{a-1}}{2 + am_b\left(m_b \frac{\sigma_3}{\sigma_{ci}} + s\right)^{a-1}} \quad (5)$$

The SSR form of analysis involves the following steps:

(i) Reduction of the shear strength envelope by a factor *F*

(ii) Determination of new strength model parameters that conform to the lowered envelope, and

(iii) Use of the new parameters in conventional FE elasto-plastic analysis.

To lower the Generalized Hoek-Brown shear strength envelope by the factor *F*, we simply divide Equation (4) by *F*.

$$\tau^{red} = \frac{\tau^{orig}}{F} = (\sigma_1 - \sigma_3)\frac{\sqrt{1 + am_b\left(m_b \frac{\sigma_3}{\sigma_{ci}} + s\right)^{a-1}}}{2 + am_b\left(m_b \frac{\sigma_3}{\sigma_{ci}} + s\right)^{a-1}} \cdot \frac{1}{F}$$
$$= (\sigma_1 - \sigma_3)\frac{\sqrt{1 + am_b^{red}\left(m_b^{red} \frac{\sigma_3}{\sigma_{ci}^{red}} + s^{red}\right)^{a^{red}-1}}}{2 + am_b^{red}\left(m_b^{red} \frac{\sigma_3}{\sigma_{ci}^{red}} + s^{red}\right)^{a^{red}-1}} \quad (6)$$

The geometric interpretation of lowering the shear envelope by a factor is illustrated on Figure 1. It shows the original shear envelope, τ^{orig}, which has been reduced by *F* to produce a reduced envelope τ^{red}.

In the above equation, σ_{ci}^{red}, m_b^{red}, s^{red}, and a^{red}, are the parameters of the lowered (reduced) strength envelope. σ_1 can be eliminated from the equation by replacing it with Equation (1).

It is important to note that given a set of Generalized Hoek-Brown parameters, and a specified σ_3 value, σ_n can be determined from Equation (5) through replacement of σ_1 with the definition of the criterion (Equation (1)).

Examination of Equation (6) reveals that the determination of the parameters of the reduced envelope is not trivial. In a previous paper [5], the authors presented a simplified approach that approximated the reduced Generalized Hoek-Brown shear envelope with a linear Mohr-Coulomb equivalent. Although a useful first approximation, this approach was quite sensitive to the range of normal stresses over which the linear approximation was determined. A new approach, which actually determines a best-fit Generalized Hoek-Brown strength model to the reduced shear envelope, is described next.

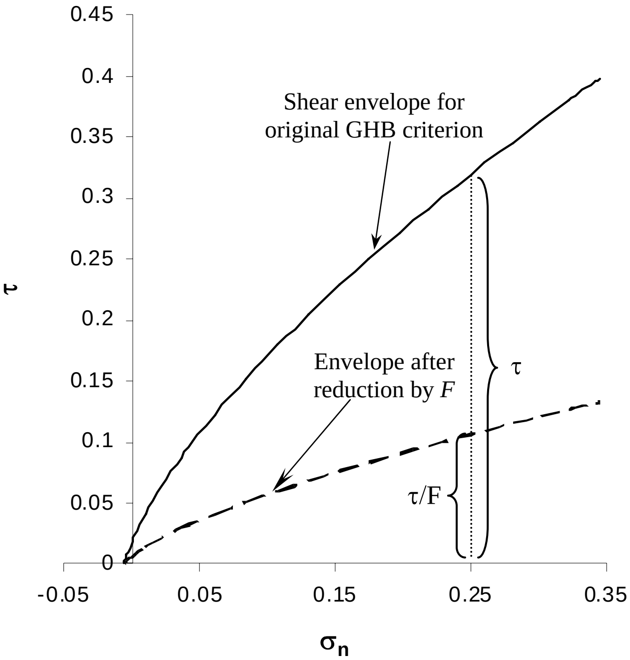

Figure 1. A Generalized Hoek-Brown criterion drawn in shear-normal stress space, and the resulting curve when the envelope is reduced by a factor *F*.

3.1. *Estimating the parameters of the reduced shear strength envelope*

Figure 2 shows the reduced shear envelope, τ^{red}, and a new Generalized Hoek-Brown, τ^{appr}, that approximates the reduced envelope. For any given σ_n value, the square of the error between the reduced and approximated envelopes is defined by the equation

$$\varepsilon(\sigma_n)^2 = \left(\tau^{appr} - \tau^{red}\right)^2 . \qquad (7)$$

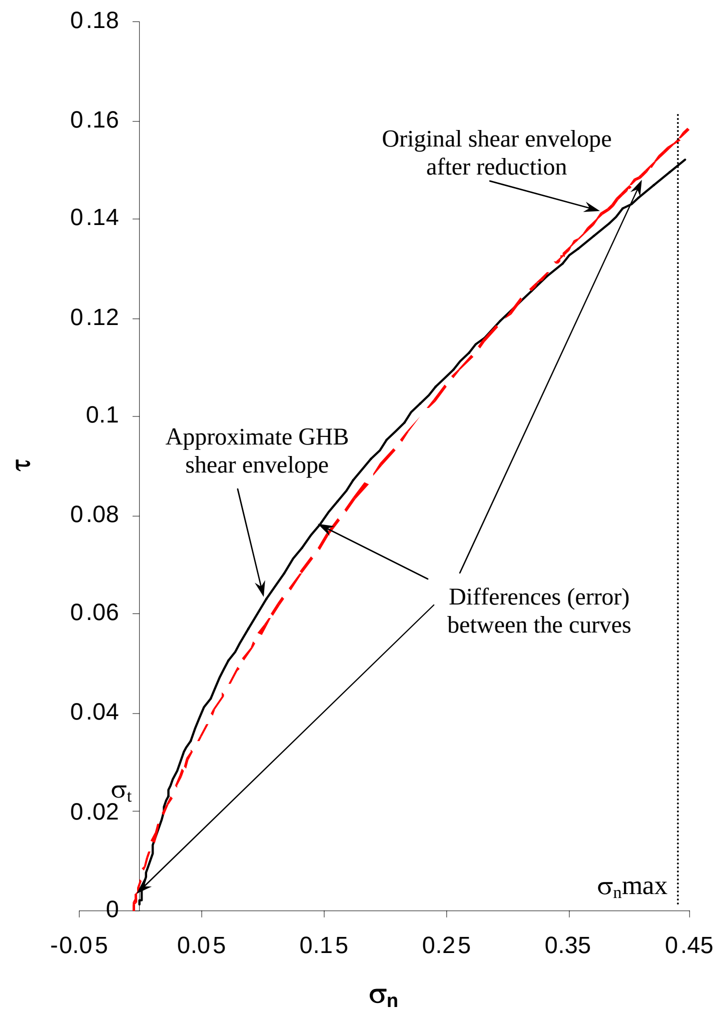

Figure 2. Reduction of a Generalized Hoek-Brown shear envelope by a factor results in a lowered curve that generally can no longer be described by a Hoek-Brown curve. The figure shows an approximation of the reduced curve with an equivalent Hoek-Brown. Notice the regions of error or differences between the two curves.

The total error of the fit of τ^{appr} to τ^{red} can be obtained through integration of the squared error function

$$Total\ error = \int_{\sigma_t}^{\sigma_n \max} \varepsilon(\sigma_n)^2 \, d\sigma_n \qquad (8)$$

over the range σ_t (the tensile strength) to a maximum normal stress value, $\sigma_n \max$. Because the squared error function does not explicitly relate σ_n

to τ, the integration is best performed using a numerical approach such as gaussian quadrature.

The parameters of the best-fit Generalized Hoek-Brown envelope to the reduced shear strength envelope can be obtained through minimization of the total squared error. This minimization is best attained with techniques such as the Simplex method, which do not require derivatives of the function being minimized.

3.2. *Algorithm for computing reduced Generalized Hoek-Brown parameters*

Based on the discussions above, the authors developed an algorithm for determining the parameters of a curve that best fit a Generalized Hoek-Brown shear strength envelope, which has been reduced by a factor, *F*. To reduce the number of parameters to be determined, it is assumed that the uniaxial compressive strength, σ_{ci}^{red}, of the reduced curve can be simply calculated as

$$\sigma_{ci}^{red} = \frac{\sigma_{ci}}{F}. \quad (9)$$

This assumption simplifies curve-fitting procedures considerably, but introduces practically no additional error.

Next, instead of directly fitting for the parameters m_b^{red}, s^{red}, and a^{red}, the procedure assumes the disturbance parameter $D = 0$, and estimates values for m_i and *GSI*. (As in the case of σ_{ci}^{red}, assuming $D = 0$ simplifies calculations substantially at very minimal penalty to accuracy.) These are then used to calculate values for m_b^{red}, s^{red}, and a^{red}.

The steps for estimating the Generalized Hoek-Brown parameters of the reduced shear envelope are then as follow:

(i) Establish the range of minor principal stresses acting in a slope. Since the minimum stress is taken to be the tensile strength, σ_t, it is only necessary to determine the maximum σ_3 value in the slope.

(ii) Determine the corresponding value of normal stress, $\sigma_n \max$, using Equation (5).

(iii) Minimize the squared error function over the range $[\sigma_t, \sigma_n \max]$ (integration of which is performed numerically) using a technique such as the Simplex method. The variables of the function are m_i and *GSI*. σ_{ci}^{red} and *D* have the fixed values described above.

4. EXAMPLES

We illustrate the capabilities of the above-outlined FE SSR technique for the Generalized Hoek-Brown criterion on two examples. The technique was implemented in the finite element program Phase2 [6]. The particular implementation tested in this paper assumes elastic-fully plastic material behaviour. (This condition can be easily relaxed though, and in a future paper the authors will discuss the impact of different elasto-plastic assumptions on results.)

For Example 1, we compare the factor of safety value computed by the FE SSR to those obtained from the equivalent Mohr-Coulomb approach described in [5], and from conventional limit-equilibrium analysis [7]. In the second example we compare the FE SSR result to values computed from limit-equilibrium analysis.

4.1. *Example 1*

Example 1 involves analysis of a 10 m high homogeneous rock slope with a 45^o slope angle (Figure 1). The Generalized Hoek-Brown parameters of the slope rock mass are provided in Table 1. Stresses in the slope are assumed gravitational, with a horizontal to vertical stress ratio of 1.

Table 1. Properties of the rock mass in the Example 1slope

Property	**Value**
Young's modulus, E (MPa)	5000
Poisson's ratio, ν	0.3
Weight, γ (MN/m^3)	0.025
Uniaxial compressive strength σ_{ci} (MPa)	30
GSI	5
Intact rock parameter m_i	2
Disturbance factor, *D*	0
Parameter m_b	0.067
Parameter *s*	2.5 x 10^{-5}
Parameter *a*	0.619

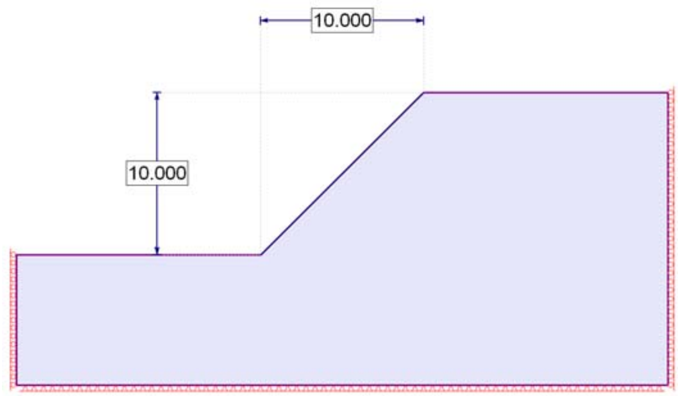

Figure 3. Geometry of the slope in Example 1.

Table 2 below shows the factor of safety result obtained from the FE SSR method, and compares it to that calculated by the equivalent Mohr-Coulomb approximation described in [5]. It also shows the factor of safety values calculated by the Bishop simplified and Spencer limit-equilibrium methods.

The SSR method for the Generalized Hoek-Brown strength model gave near identical results to those given by the other methods. Its predictions of the failure mechanism (slip surface) are shown on Figure 3.

Table 2. Comparison of factor of safety results for Example 1.

Method	**Factor of Safety**
FE SSR technique:	
Generalized Hoek-Brown	1.15
Equivalent Mohr-Coulomb	1.15
Limit equilibrium:	
Bishop's simplified	1.153
Spencer's method	1.152

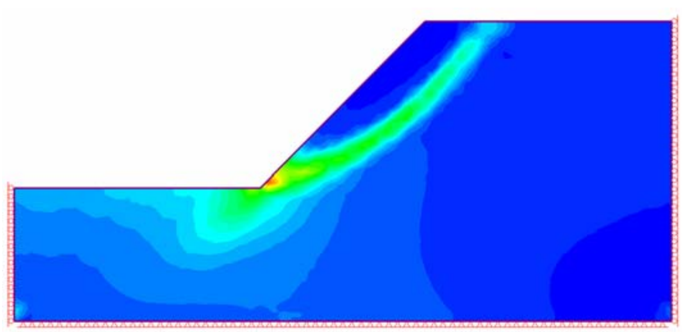

Figure 4. Contours of maximum shear strain in the slope at failure. The contours reveal the failure mechanism predicted by the SSR method.

4.2. Example 2

Example 2 was selected to test the performance of the proposed approach when a material of a different strength type is also present in a slope. The slope in Example 1, but this time with a horizontal layer of Mohr-Coulomb material passing through the toe (shown in Figure 5), was analyzed. The Mohr-Coulomb layer had zero cohesion, a 25^{o} friction angle, and a thickness of 1m.

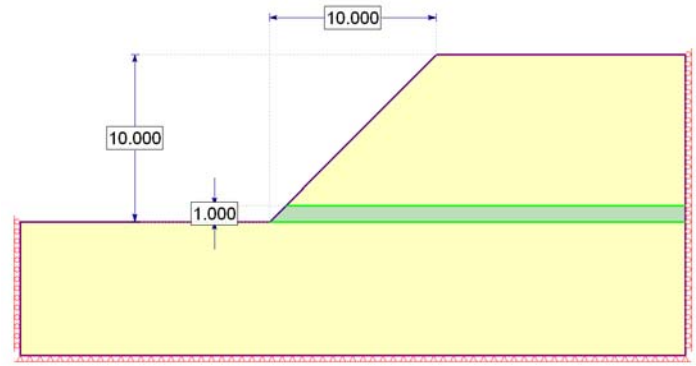

Figure 5. Geometry of slope in Example 2.

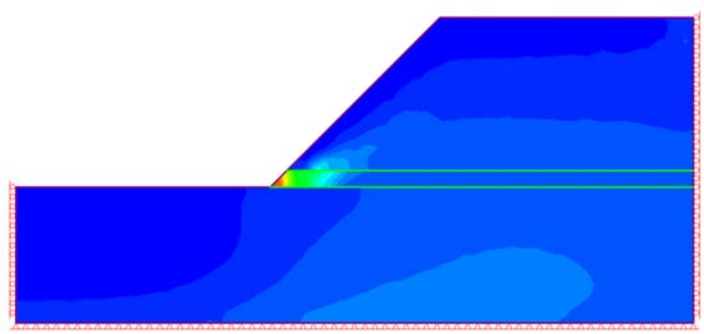

Figure 6. Contours of maximum shear strain in Example 2 slope. Note the failure mechanism passing through the toe of the slope.

The factor of safety value obtained from SSR analysis of the slope is compared to those of non-circular Bishop simplified and Spencer limit-equilibrium analysis in Table 3. Again the SSR method for the Generalized Hoek-Brown is very close to those computed from limit-equilibrium analysis. Also the failure mechanism indicated by the SSR method, which is shown on Figure 6, is as expected.

Table 3. Comparison of factor of safety results for Example 2.

Method	**Factor of Safety**
FE SSR technique:	
Generalized Hoek-Brown	0.95
Limit equilibrium:	
Bishop's simplified	0.934
Spencer's method	0.963

5. CONCLUSION

The development of an SSR framework that allows direct modeling of Generalized Hoek-Brown materials allows the rock engineering community to more fully exploit the advantages and power of finite element analysis.

The FE SSR method is a robust alternative to limit-equilibrium slope stability methods. It is particularly beneficial in situations in which stress has a dominant influence on stability. It is able to calculate the deformations of reinforcement elements such as rock bolts, anchors, and piles, as well as important quantities such as bending moments.

Like limit-equilibrium methods, the SSR technique can accommodate multiple material layers, phreatic surfaces and seepage results. Unlike its limit-equilibrium counterparts, it does not require *a priori* assumptions on failure mechanisms (the shapes of failure surfaces).

When contour plots of stresses and displacements, such as those shown in this paper, are arranged in sequence (for ordered factor of safety values ranging from stable to unstable) they provide insightful information on the development of failure mechanisms.

In addition to allowing rock engineers to harness the above-listed advantages, the Generalized Hoek-Brown formulation of the SSR technique makes it possible to analyze limit-equilibrium slope models involving non-linear material strength envelopes such as the power curve criterion. Such strength criteria, popular for soil materials, are difficult to use in elasto-plastic finite element analysis due to the absence of flow rules.

Using techniques similar to that used to fit a Generalized Hoek-Brown curve to a reduced shear envelope, equivalent Hoek-Brown parameters can be determined for a non-linear strength envelope. SSR analysis can then be performed on a slope model using the equivalent Generalized Hoek-Brown envelope.

Given the benefits of the SSR technique, the hope authors hope that it will be applied more frequently to rock slope problems.

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