Theory Manual

Boundary Element Method (BEM) is one of the well-known numerical methods to solve the differential equations. This approach has been used in a variety of fields such as electromagnetics, solid mechanics, fracture mechanics, fluid mechanics and acoustics. The main benefit of BEM is reducing one-degree dimensionality of the problem, i.e. instead of discretizing a 3D volume, the discretization would only be assigned to the surface of the volume.

In solid mechanics, in order to solve the equilibrium equation in elastic domain with BEM, two approaches can be used, direct and indirect method. With the *direct* method, by employing an appropriate Dirac-Delta function to the weak form of equation and using divergence theory, the equilibrium equation can be solved. However, the *indirec*t formulation uses a singular solution that satisfies the governing equations over the boundaries of interest, such as the free-space Green functions (Banerjee and Butterfield, 1981). Noting that, EX3 is using the indirect method.

Formulation

Considering a homogeneous isotropic linear elastic domain subjected to a concentrated load ***e**,* at point $\boldsymbol{x}_q$, the deformation at any interest point $\boldsymbol{x}_p$ can be written as:

$$u_i(x_p) = B_{ij}(x_p, x_q) e_j(x_q)$$

Where B_{ij} is the second order tensor and the components of this tensor can be fined at references. (Banerjee and Butterfield, 1981). Knowing the displacement vector, the strain and stress can be derived as following:

$$\varepsilon_{ij}(x_p) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = H_{ijk}(x_p, x_q) e_k(x_q)$$

Where, ε_{ij} is the strain tensor. By applying Hook's law, one can write:

$$\sigma_{ij}(x_p) = D_{ijkl}\, \varepsilon_{kl}(x_q) = G_{ijk}(x_p, x_q) e_k(x_q)$$

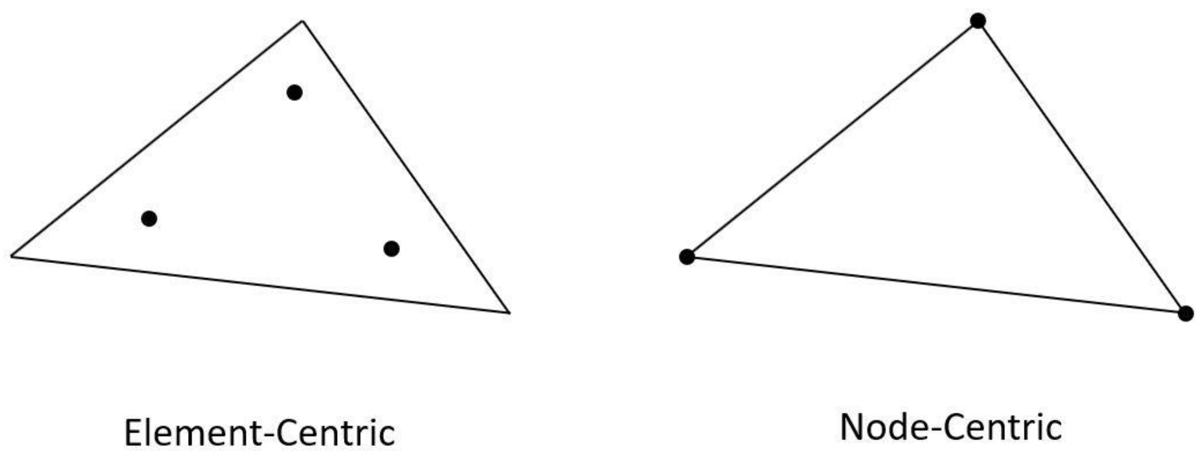
Here σ_{ij} is the stress tensor, D_{ijkl} is the forth-order elastic constitutive tensor and G_{ijk} is the third order tensor associated to the green functions (Crouch and Starfield, 1983). By knowing the stress at any interest point, the traction on the boundary surface can be calculated as:

$$t_i(x_p) = \sigma_{ij}(x_p).n_j = K_{ij}(x_p, x_q) e_j(x_q)$$

The previous relation is the basic equation to be solved for any geometry. For more information, please read (Yacoub 1998).

Node-Centric method

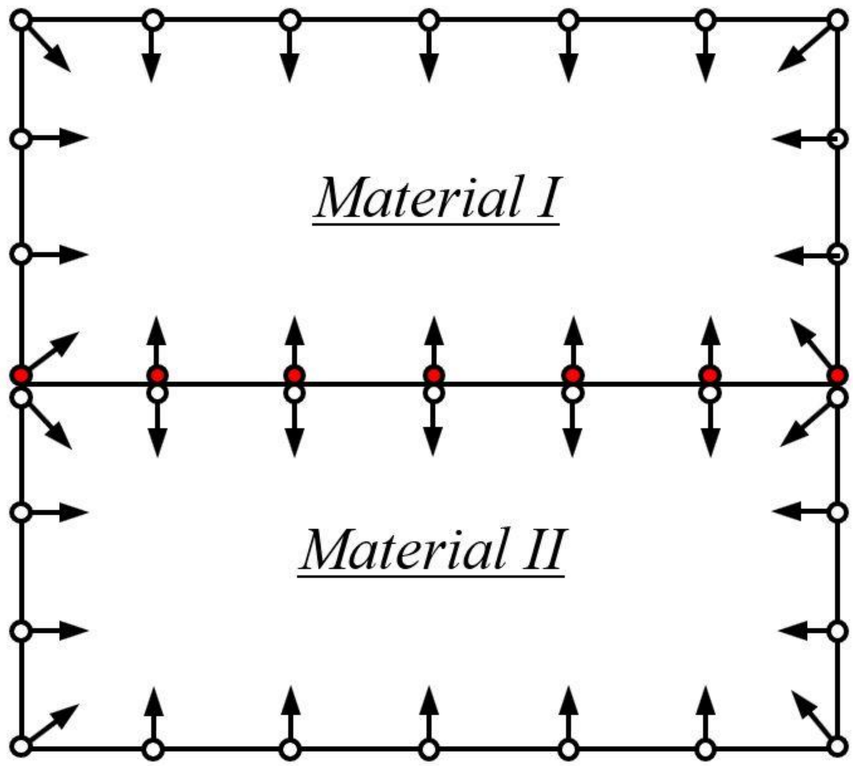
In order to solve the equilibrium equation, the boundaries can be discretized using triangular and quadrilateral elements. One of the earliest approaches for integration on the surface defined by using constant element without node sharing. In this discretization, despite finite element mesh, the nodes are not connected to adjacent elements. The main reason of developing this technique is to eliminate the singularities that occur on the corners. However, this can lead to higher number of nodes which reduces the speed of calculation. To resolve this issue, Yacoub 1998, proposed a new method called the node-centric method. In spite of constant elements, here nodes are shared with adjacent element and the normal of each node is calculated by the average of the neighbor element's normal.



In general, boundary element integrals can be classified into two categories: singular and non-singular cases. When the subject node $\boldsymbol{x}_p$ and interest point $\boldsymbol{x}_q$ are very close to each other, the integrals can become unbounded. Vijaykumar and Cormack, 1989, introduced a new integration methodology to resolve the singularity. In this technique, when the singular or near singular cases happen, the integration is taken over the edges of elements. Otherwise, the standard integration on the surface would be applied. This approach is implemented in EX3 to reduce the singularity cases.

Multi-Material

One of the new features of EX3 is modeling multi-material domains. To explain the methodology, let's consider the following domain containing two materials zones.



Each zone would be discretized where the normal direction of each element is facing inward. The basic assumption here is the continuity on the material boundaries, i.e., the total displacement should be the same on each boundary. In addition, due to equilibrium, the summation of tractions on the boundary surfaces is zero:

$$u_i^m - u_i^s = 0, \quad t_i^m - t_i^s = 0$$

Here, superscript *m*, stands for master node on the material boundary I and *s,* devoted to the slave node associated to material boundary II. By combining the above equations in the equilibrium equation, the multiple material domains can be solved with BEM. For more information please read (Moallemi et.al., 2019).

Joint

Modeling discontinuities in EX3 is implemented using displacement discontinuity method (DDM). This methodology is based on an analytical solution of a constant discontinuity over a crack segment in an infinite domain. This discontinuity can be interpreted as sliding or opening of a crack. Despite Finite Element Method defining that a joint requires double layer elements, here in BEM, only one layer of elements is needed to define the crack. The green function associated to DDM with node-centric method are defined by Vijaykumar et.al. 2000.

Reference

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