1- Preliminaries on Constitutive Models

Constitutive modeling or constitutive equations are the mathematical relationship between stress and strain for a material. This introductory section summarizes the basic definitions and concepts that are used in mechanical constitutive models in RS^2 and RS^3. The specific constitutive models will be presented in the following chapters. To benefit the users, the formulation of each model is presented briefly and the applicability of the models are demonstrated by examples and verifications.

1.1- Stress

In continuum mechanics, stress is a physical quantity that expresses the internal forces that neighboring particles of a continuous material exert on each other. Stress is defined as the average force per unit area that some particle of a body exerts on an adjacent particle, across an imaginary surface that separates them (e.g. Pilkey & Pilkey 1974, Daintith 2005, Josephs 2009).

State of stress is defined by nine components, six of which are independent. These 6 components form the stress tensor (Figure 1.1).

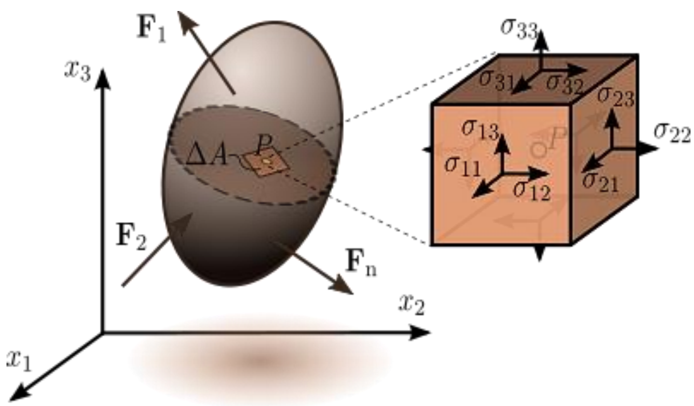

Figure 1.1 - Stress state at a point in a body under loads

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad ; \quad \sigma_{12} = \sigma_{21} \,;\, \sigma_{13} = \sigma_{31} \,;\, \sigma_{32} = \sigma_{23} \tag{1.1}$$

Stress state can also be defined by specifying stress vectors associated with base vectors e_i (Pietruszczak 2010).

$$t_i^{(e_i)} = \sigma_{ij} e_j \quad ; \quad e_i = e_1 \,, e_2 \,, e_3 \tag{1.2}$$

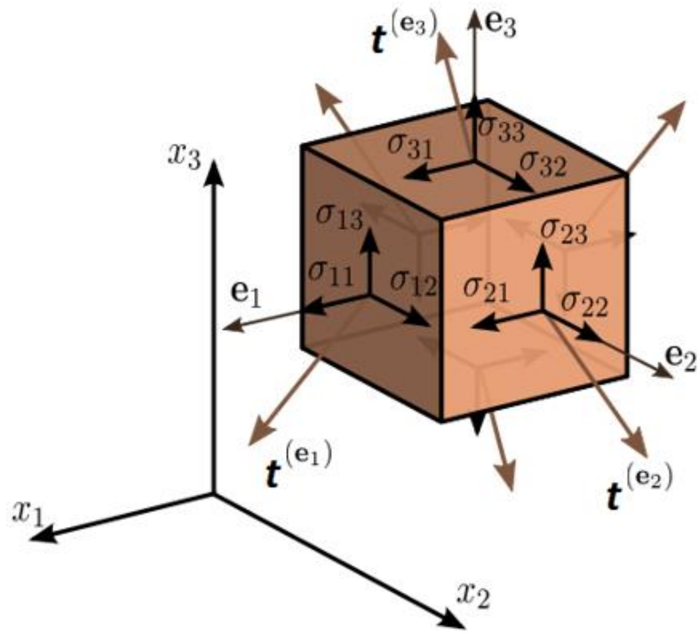

Figure 1.2 - Stress state and vectors at a point

Considering a transformation of a coordinate system consisting of a rotation about an axis, the transformed stress state in the new coordinate system can be obtained from equation (1.3).

$$\sigma_{ij}^{*} = T_{ip}\sigma_{pq}T_{jq} \tag{1.3}$$

where T_{ij} is the transformation/rotation matrix with the components that are the direction cosines of the new base vectors l_i , m_i and n_i .

$$T_{ij} = \begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} \tag{1.4}$$

Any object which transforms according to equation (1.3) is referred to as a second order tensor, thus the stress state represent a symmetric second order tensor.

Stress can also be represented in a vector form. This could be helpful in calculations and computer programming. The stress vector has the same six independent components:

$$\underline{\sigma} = (\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{13})^T \tag{1.5}$$

One should note that the stress-strain relationship for geomaterials and in general porous media is formulated in terms of effective stress. In this manual, the stresses that appear in all the formulations are effective stresses.

1.1.1- Principal Stresses and Stress Invariants

One can always find an orientation of the coordinate system that the stress vectors are collinear with base vectors. Such directions are referred to as principal directions (e.g. Pietruszczak 2010).

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$$\begin{gathered} t_i^{(v_i)} = \sigma_{ij} v_j = \sigma v_j \\ \sigma_{ij} v_j = \sigma \delta_{ij} v_j \\ (\sigma_{ij} - \sigma \delta_{ij})\, v_j = 0 \end{gathered} \tag{1.6}$$

This represents an eigenvalue problem where σ's are the eigenvalues and v_i are the eigenvectors of σ_{ij}. The solution requires that

$$det(\sigma_{ij} - \sigma \delta_{ij}) = 0 \tag{1.7}$$

This determinant gives a cubic equation known as the characteristic equation

$$\sigma^3 - I_1 \sigma^2 - I_2 \sigma - I_3 = 0 \tag{1.8}$$

where

$$\begin{gathered} I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \\ I_2 = -\frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right) = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 \\ I_3 = \det[\sigma_{ii}] \end{gathered} \tag{1.9}$$

Since the stress tensor is symmetric there are always three real eigenvalues, in other words equation (1.7) always has three roots which represent the principal stresses.

Associated with each principal stress there is a principal stress direction which can be uniquely determined. Since the stress tensor is symmetric, principal stress directions are orthogonal.

The values of principal stresses are independent of the coordinate system. This means that the I's are invariants with respect to the choice of a frame of reference. If the axes are chosen to coincide with the principal axes then one can conclude

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3\ ;\ \ I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\ ;\ \ I_3 = \sigma_1\sigma_2\sigma_3 \tag{1.9}$$

The following decomposition of stress is often useful, as will be seen in later chapters:

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}I_1 \tag{1.10}$$

s_{ij} is called the stress deviator. The stress deviator is also a second order tensor and its principal values are defined from the characteristic equation:

$$s^3 - J_2 \mathrm{s} - J_3 = 0 \tag{1.11}$$

where

$$J_1 = s_{ii} = 0\ \ ;\ \ J_2 = \tfrac{1}{2}s_{ij}s_{ij} = (I_1^2 + 3I_2)\ ;\ \ J_3 = \tfrac{1}{3}s_{ij}s_{jk}s_{ki} = (2I_1^3 + 9I_1I_2 + 27I_3)/27 \tag{1.12}$$

In addition to these stress and stress deviator invariants the following invariants are also useful:

$$p = \frac{I_1}{3}$$

$$q = \sqrt{3J_2} = \sqrt{\tfrac{1}{2}\left((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6 * (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\right)} \tag{1.13}$$

$$\theta = \frac{1}{3} arcsin\left(-\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right)$$

In above p is the mean stress, q is the deviatoric stress, and θ is Lode's angle. Using the definitions above one can find the principal stresses to be:

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\end{Bmatrix}=\frac{2\sqrt{J_2}}{\sqrt{3}}\begin{Bmatrix}sin\left(\theta+\frac{2\pi}{3}\right)\\sin\theta\\sin\left(\theta+\frac{4\pi}{3}\right)\end{Bmatrix}+\frac{J_1}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix} \tag{1.14}$$

1.2- Strain

In continuum mechanics, the Cauchy strain or engineering strain is expressed as the ratio of total deformation to the initial dimension of the material body in which the forces are being applied. The components of infinitesimal strain tensor(ε_{ij}), also called Cauchy's strain tensor, linear strain tensor, or small strain tensor are derivatives of displacement components (e.g. Pilkey & Pilkey 1974, Daintith 2005, Josephs 2009):

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)=\begin{pmatrix}\varepsilon_{11}&\varepsilon_{12}&\varepsilon_{13}\\\varepsilon_{21}&\varepsilon_{22}&\varepsilon_{23}\\\varepsilon_{31}&\varepsilon_{32}&\varepsilon_{33}\end{pmatrix} \tag{1.15}$$

where u_i is the displacement vector and index "," is derivative in the direction of j.

The strain can be represented in a vector form as well

$$\underline{\varepsilon}=(\varepsilon_{11}\quad\varepsilon_{22}\quad\varepsilon_{33}\quad\gamma_{12}\quad\gamma_{23}\quad\gamma_{13})^T \tag{1.16}$$

In above $\gamma_{ij}=\varepsilon_{ij}+\varepsilon_{ji}$

Similar to the stress tensor strain tensor has its own principal values and principal directions. Strain deviator, e_{ij} , is also commonly used in constitutive equations. The most useful invariants of strain tensor are the volumetric strain, ε_v , and deviatoric strain, ε_q. Another measure of the deviatoric strain is γ that has been used in formulation of constitutive equations.

$$e_{ij}=\varepsilon_{ij}-\frac{1}{3}\delta_{ij}\varepsilon_{kk} \tag{1.17}$$

$$\varepsilon_v=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\quad;\quad\varepsilon_q=\sqrt{\frac{2}{3}e_{ij}e_{ij}}\quad;\quad\gamma=\sqrt{e_{ij}e_{ij}} \tag{1.19}$$

References

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