2- Elastic Material Models

The general equation for the elastic constitutive models in incremental form is

$$\sigma_{ij} = D^{e}_{ijkl}\varepsilon_{kl} \tag{2.1}$$

In above D^{e}_{ijkl} is the fourth order elastic constitutive tensor. The elastic constitutive equations can also be presented using stress and strain vectors:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{2.2}$$

In above, E is the elastic modulus and ν is the Poisson's ratio.

Other elastic properties that could be useful are the shear modulus, G, and bulk modulus K.

$$G = \frac{E}{2(1+\nu)} \quad ; \quad K = \frac{E}{3(1-2\nu)}$$

The elastic behavior can be linear or nonlinear. In linear elasticity the elastic material properties are constants, but in nonlinear elastic models they change based on some assumptions. There are different options for linear and nonlinear elasticity in RS^2 and RS^3, the following sections will present all these options. There are also a few constitutive models that have their own specific formulation for elastic behavior. For those special cases the elastic behavior is discussed in their corresponding sections.

2.1- Linear Elasticity

In linear elastic material models, the elastic properties are constants. The options in these family of models are isotropic, transversely isotropic and orthotropic elastic behavior.

2.1.1- Linear Isotropic Elastic Model

The constitutive equations for linear isotropic elastic behavior is presented in equation (2.2).

This equation is for the three dimensional situations as in RS[3]. RS2 is a 2 dimensional program and it is limited to plane-strain and axisymmetric configurations. For the reference, the elastic constitutive equations for plane-strain and axisymmetric cases are presented in equation (2.3).

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \sigma_{33}\end{Bmatrix} = \frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix}1-\nu & \nu & 0 & \nu\\ \nu & 1-\nu & 0 & \nu\\ 0 & 0 & (1-2\nu)/2 & 0\\ \nu & \nu & 0 & 1-\nu\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\\ \varepsilon_{33}\end{Bmatrix} \tag{2.3}$$

2.1.2- Linear Transversely Isotropic Elastic Model

The constitutive equations for linear transversely isotropic elastic material in the material coordinated, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is presented in equation (2.4). It is assumed that $\hat{e}_3$ is the unit vector normal to the plane of isotropy.

$$\hat{\sigma}_{ij} = \widehat{D}^e_{ijkl}\hat{\varepsilon}_{kl} \tag{2.4}$$

The constitutive equation in the material coordinate system can be formulated in terms of 5 independent elastic parameters, $E_1, E_3, \nu_{12}, \nu_{13}, G_{13}$ (e.g. Timoshenko and Goodier 1970, Saada 1993)

The components of the transversely isotropic constitutive tensor are:

$$\begin{aligned}
&\hat{D}^e_{1111} = \hat{D}^e_{2222} = E_1(1-\nu_{31}\nu_{13})\Upsilon\\
&\hat{D}^e_{3333} = E_3(1-\nu_{12}^2)\Upsilon\\
&\hat{D}^e_{1122} = \hat{D}^e_{2211} = E_1(\nu_1+\nu_{31}\nu_{13})\Upsilon\\
&\hat{D}^e_{1133} = \hat{D}^e_{3311} = \hat{D}^e_{2233} = \hat{D}^e_{3322} = E_1(\nu_{31}+\nu_{31}\nu_{12})\Upsilon = E_3(\nu_{13}+\nu_{13}\nu_{12})\Upsilon\\
&\hat{D}^e_{1212} = \hat{D}^e_{2121} = \hat{D}^e_{1221} = \hat{D}^e_{2112} = G_{12}\\
&\hat{D}^e_{1313} = \hat{D}^e_{3131} = \hat{D}^e_{1331} = \hat{D}^e_{3113} = G_{13}\\
&\hat{D}^e_{2323} = \hat{D}^e_{3232} = \hat{D}^e_{2332} = \hat{D}^e_{3223} = G_{13}
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&\Upsilon = (1-\nu_{12}^2 - 2\nu_{13}\nu_{31} - 2\nu_{12}\nu_{13}\nu_{31})^{-1}\\
&G_{12} = \frac{1}{2}\left(D_{1111} - D_{1122}\right) = \frac{E_1}{2(1+\nu_{12})}\\
&E_1\,\nu_{31} = E_3\,\nu_{13}
\end{aligned} \tag{2.6}$$

In RS^2 and RS^3, for using this option of constitutive equations the increment of strain is first transformed from the global coordinate to the material coordinate. The increment of stress is then calculated based on equation 2.4 in the material coordinate, and finally the increment of stress is transformed from the material coordinate to the global coordinate and added to the state of stress in global coordinate.

2.1.3- Linear Orthotropic Elastic Model

The constitutive equations for linear transversely isotropic elastic material in the material coordinated, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is presented in equation (2.7).

$$\hat{\sigma}_{ij} = \widehat{D}^{e}_{ijkl}\hat{\varepsilon}_{kl} \tag{2.7}$$

The constitutive equation in the material coordinate system can be formulated in terms of 9 independent elastic parameters, $E_1, E_2, E_3, \nu_{12}, \nu_{13}, \nu_{23}, G_{12}, G_{13}, G_{23}$.

The components of the transversely isotropic constitutive tensor are:

$$\begin{aligned}
&\hat{D}^e_{1111} = E_1(1-\nu_{23}\nu_{32})\Upsilon \\
&\hat{D}^e_{2222} = E_2(1-\nu_{31}\nu_{13})\Upsilon \\
&\hat{D}^e_{3333} = E_3(1-\nu_{21}\nu_{12})\Upsilon \\
&\hat{D}^e_{1122} = \hat{D}^e_{2211} = E_1(\nu_{21}+\nu_{31}\nu_{23})\Upsilon = E_2(\nu_{12}+\nu_{13}\nu_{32})\Upsilon \\
&\hat{D}^e_{1133} = \hat{D}^e_{3311} = E_1(\nu_{31}+\nu_{21}\nu_{32})\Upsilon = E_3(\nu_{13}+\nu_{12}\nu_{23})\Upsilon \\
&\hat{D}^e_{2233} = \hat{D}^e_{3322} = E_2(\nu_{32}+\nu_{12}\nu_{31})\Upsilon = E_3(\nu_{23}+\nu_{21}\nu_{13})\Upsilon \\
&\hat{D}^e_{1212} = \hat{D}^e_{2121} = \hat{D}^e_{1221} = \hat{D}^e_{2112} = G_{12} \\
&\hat{D}^e_{1313} = \hat{D}^e_{3131} = \hat{D}^e_{1331} = \hat{D}^e_{3113} = G_{13} \\
&\hat{D}^e_{2323} = \hat{D}^e_{3232} = \hat{D}^e_{2332} = \hat{D}^e_{3223} = G_{13}
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
&\Upsilon = (1-\nu_{12}\nu_{21}-\nu_{13}\nu_{31}-\nu_{23}\nu_{32}-2\nu_{21}\nu_{32}\nu_{13})^{-1} \\
&E_i\,\nu_{ji} = E_j\,\nu_{ij}
\end{aligned} \tag{2.9}$$

In RS^2 and RS^3, for using this option of constitutive equations the increment of strain is first transformed from the global coordinate to the material coordinate. The increment of stress is then calculated based on equation 2.4 in the material coordinate, and finally the increment of stress is transformed from the material coordinate to the global coordinate and added to the state of stress in global coordinate.

2.2- Nonlinear Elasticity

In nonlinear elastic material models the elastic properties are dependent on some measures of stress or strain tensor. The nonlinear elasticity in RS^2 and RS^3 is isotropic.

The first two options for nonlinear elasticity is based on the fact that the elastic modulus of porous materials is proportional to the confinement.

$$E = E_0 \left(\frac{p}{p_{ref}}\right)^{\alpha} \tag{2.10}$$

$$E = E_0 \left(\frac{bp+a}{p_{ref}+a}\right)^{\alpha} \tag{2.11}$$

In above E_0 is the elastic modulus at reference pressure. p_{ref} is the reference pressure, and a, b and α are material parameters. p is the mean stress, assuming compression positive. The value of α is usually between 0.5 and 1.0.

Laboratory data suggests degradation of shear modulus with increase in the deviatoric strain. To capture such an effect there is another option for nonlinear elasticity in the RS^2 and RS^3

$$G = G_{max} \left(1 + a\frac{\gamma}{\gamma_y}\right)^{r} \tag{2.12}$$

In above G_{max} is the maximum shear modulus and, a, γ_y and r are material parameters. To simulate degradation parameter r should be less than zero. The deviatoric strain, γ, is reset to zero every time the direction of loading changes.

To combine the effect of confinement and deviatoric strain, one can use the combination of equations (2.13) and (2.14). In the first equation the maximum elastic modulus is calculated based on the level of confinement and in the second one the degradation because of deviatoric strain is taken into account.

$$E_{max} = E_0 \left(\frac{bp+a}{p_{ref}+a}\right)^{\alpha} \tag{2.13}$$

$$E = E_{max} \left(1 + a\frac{\gamma}{\gamma_y}\right)^{r} \tag{2.14}$$

For this last case, the deviatoric strain, γ, depends on the loading history. Once the direction of loading is changed the stiffness regains a maximum recoverable value in the order of its initial value, E_{max}. When an increment of strains is applied to the material, each principal direction is checked for a possible change in the loading direction. This option can be used to mimic the hysteretic behavior of soils in dynamic loading, but it is not a robust constitutive model for this purpose, since this phenomenon is best described by using deviatoric hardening plasticity models.

The Duncan-Chang model (Duncan and Chang, 1970) is also a nonlinear elastic model that is included in RS^2 and RS^3, but his model will be presented in a separate chapter. There are other models that take advantage of nonlinear elastic behavior that is embedded in their formulations, these models will also be explained in details in in their own chapters.

2.3- Viscoelastic Material Models

Viscoelastic materials exhibit both elastic and viscous characteristics when undergoing deformation. The viscosity gives a strain rate that is time dependent. Generally, to model viscoelastic material they are assumed to be composed of springs (elasticity) and dashpots (viscosity). The arrangements of these components will give different viscoelastic constitutive models (e.g. Christensen 2012, Simo and Hughes 1980).

2.3.1- Maxwell Model

The uniaxial Maxwell model consists of a linear elastic spring and a linear viscous dashpot element connected in a series as shown in Figure 2.1.

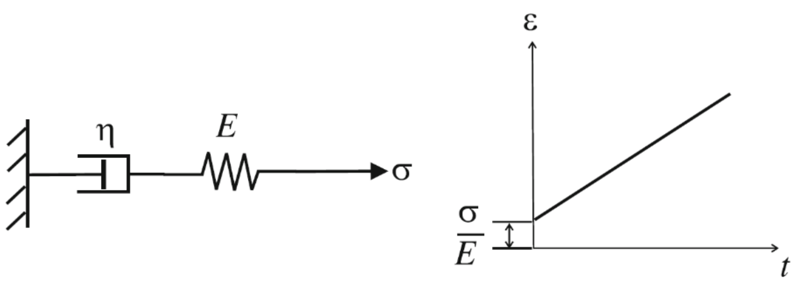


Figure 2.1- Uniaxial Maxwell material

The strain rate for this model is

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{2.155}$$

For the generalization of this model to 3D states of stress and strain it the viscous behavior is assumed to happen only for the deviatoric stress-strains and the volumetric behavior is only elastic.

The material properties to define this constitutive model are the elastic bulk modulus (K), Maxwell shear modulus (G_M) and Maxwell viscosity parameter (η_M). The deviatoric strain for the Maxwell viscoelastic material is calculated from

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2G_M} + \frac{s_{ij}}{2\eta_M} \tag{2.16}$$

Using the above equation and combining it with an elastic-only volumetric behavior the constitutive equations of Maxwell model in the incremental form are as follows

$$s_{ij}^N = \left(s_{ij}^O C_1 + 2G_M \Delta e_{ij}\right) C_2 \tag{2.17}$$

$$C_1 = 1 - \frac{G_M \Delta t}{2\eta_M} \quad , \quad C_2 = \frac{1}{1+\frac{G_M \Delta t}{2\eta_M}}$$

$$p^N = p^O + K \Delta \varepsilon_v$$

$$\sigma_{ij}^N = s_{ij}^N + \delta_{ij} p^N$$

The superscript "*O*" and "*N*" stand for the stress state at the previous stage and current stage respectively.

2.3.2- Standard Model

The uniaxial Standard model consists of a linear elastic spring and a Kelvin unit connected in a series as shown in Figure 2.2. The Kalvin unit itself, consists of a linear spring and a dashpot that are connected in parallel as shown in Figure 2.3.

Once again for the generalization to 3D states of stress and strain the viscous behavior is assumed to happen only for the deviatoric stress-strains and the volumetric behavior is only elastic.

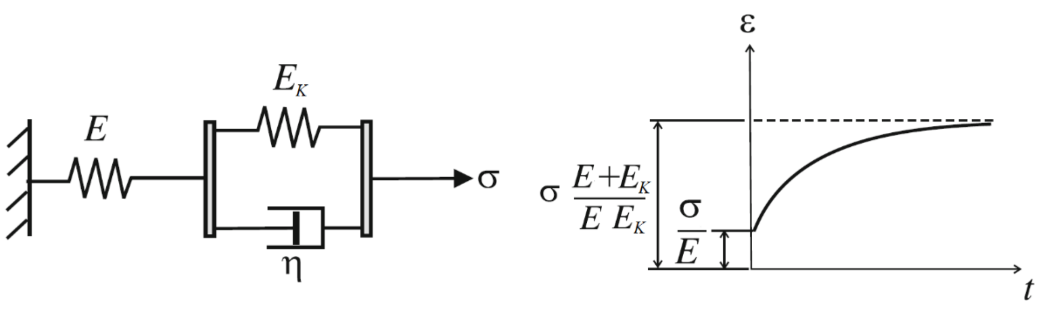


Figure 2.2- Uniaxial Standard viscoelastic model

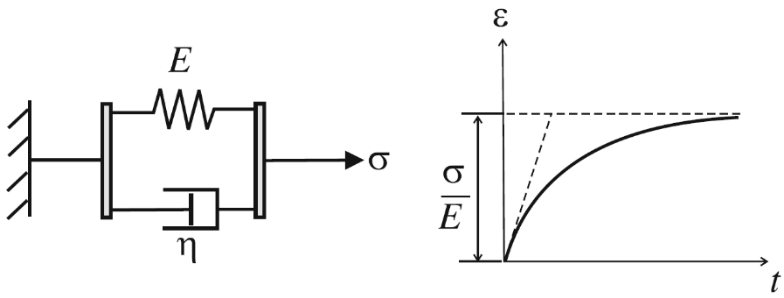


Figure 2.3- Uniaxial Kelvin model

The material properties to define this constitutive model are the elastic bulk modulus (K), elastic shear modulus (G), Kelvin shear modulus (G_K) and Kelvin viscosity parameter (η_K). The deviatoric strain is the summation of elastic deviatoric strain (spring) and the viscoelastic deviatoric strain from the Kelvin unit.

$$\dot{e}_{ij} = \left(\dot{e}_{ij}\right)_K + \left(\dot{e}_{ij}\right)_E \tag{2.18}$$

The subscript "E" represents the contribution of the (spring) elastic unit. The relationship of the deviatoric stress and Kalvin unit is:

$$s_{ij} = 2\eta_K\left(\dot{e}_{ij}\right)_K + 2G_K\left(e_{ij}\right)_K \tag{2.19}$$

Using equations (2.18) and (2.19) with an additional elastic-only volumetric behavior the constitutive equations of the Standard model in the incremental form are as follows

$$s_{ij}^N = \frac{1}{a}\left(\Delta e_{ij} + b s_{ij}^O - \left(\frac{B}{A} - 1\right) e_{ij}^{K,O}\right) C_2 \tag{2.20}$$

$$\mathrm{A} = 1 + \frac{G_K \Delta t}{2\eta_K} \quad , \quad \mathrm{B} = 1 - \frac{G_K \Delta t}{2\eta_K}$$

$$\mathrm{a} = \frac{1}{2G} + \frac{\Delta t}{4A\eta_K} \quad , \quad \mathrm{b} = \frac{1}{2G} - \frac{\Delta t}{4A\eta_K}$$

$$p^N = p^O + K\Delta\varepsilon_v$$

$$\sigma_{ij}^N = s_{ij}^N + \delta_{ij} p^N$$

2.3.3- Burgers Model

The uniaxial Burgers model consists of a Maxwell unit and a Kelvin unit connected in a series as shown in Figure 2.4.

For the generalization of this model to 3D states of stress and strain it the viscous behavior is assumed to happen only for the deviatoric stress-strains and the volumetric behavior is only elastic.

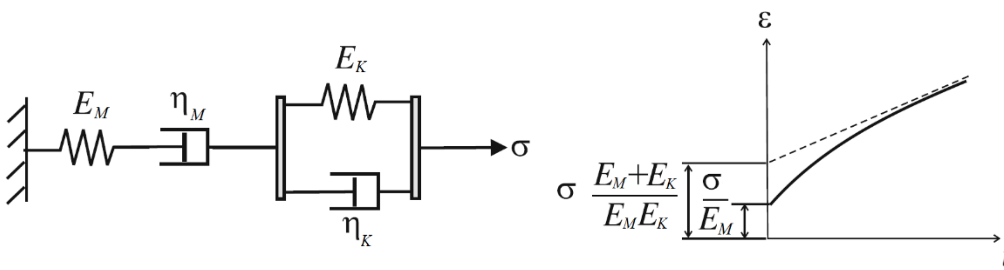


Figure 2.4- Uniaxial Burgers viscoelastic model

The material properties to define this constitutive model are the elastic bulk modulus (K), Maxwell shear modulus (G_M), Maxwell viscosity parameter (η_M), Kelvin shear modulus (G_K) and Kelvin viscosity parameter (η_K). The deviatoric strain is the summation of Maxwell deviatoric strain and Kelvin deviatoric strain.

$$\dot{e}_{ij} = \left(\dot{e}_{ij}\right)_M + \left(\dot{e}_{ij}\right)_K \tag{2.21}$$

The Maxwell and Kelvin deviatoric strain is related to the deviatoric stress as presented in equations (2.16) and (2.19) respectively. Combining those with equation (2.21) with an additional elastic-only volumetric behavior the constitutive equations of Burgers model in the incremental form are as follows

$$s_{ij}^N = \frac{1}{a}\left(\Delta e_{ij} + b s_{ij}^O - \left(\frac{B}{A} - 1\right) e_{ij}^{K,O}\right) C_2 \tag{2.22}$$

$$\mathrm{A} = 1 + \frac{G_K \Delta t}{2\eta_K} \quad , \quad \mathrm{B} = 1 - \frac{G_K \Delta t}{2\eta_K}$$

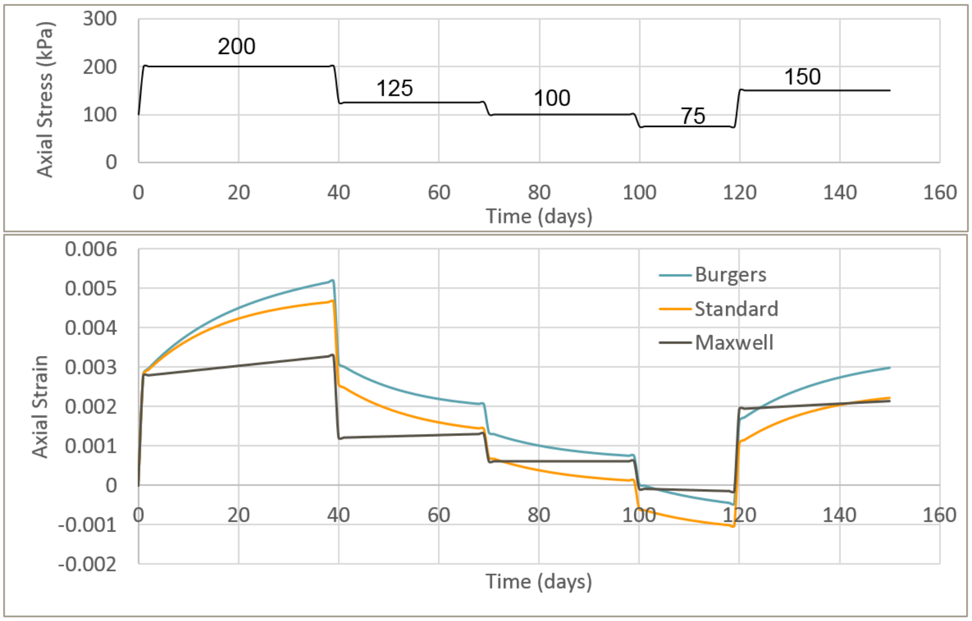
$$\mathrm{a} = \frac{1}{2G_M} + \frac{\Delta t}{4}\left(\frac{1}{\eta_M} + \frac{1}{A\eta_K}\right) \quad , \quad \mathrm{b} = \frac{1}{2G_M} - \frac{\Delta t}{4}\left(\frac{1}{\eta_M} + \frac{1}{A\eta_K}\right)$$

$$p^N = p^O + K\Delta\varepsilon_v$$

$$\sigma_{ij}^N = s_{ij}^N + \delta_{ij} p^N$$

2.3.4- Examples

Figure 2.4 shows the predicted behavior of the viscoelastic models presented above in simulations of a triaxial test that includes time dependent loading unloading and periods of rest in its loading history. The initial confinement is 100kPa and the variation of axial stress with time is presented on top of the stress path. The material properties used in the simulation are also presented in the same figure.



G_K = 16 MPa

η_K = 500 MPa*day

G_M = 16 MPa

η_M = 500 MPa*day

K = 16 MPa

G = 16 MPa

Figure 2.5- Drained triaxial compression test with loading-unloading on viscoelastic models

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