rocscience

Convergence Criteria

Finite Element Analysis of Static Problems

For static finite element analysis, the equation representing equilibrium can be written in the following matrix form:

$$\mathbf{K}\Delta\mathbf{U} = \mathbf{P} - \mathbf{F},$$

where **P** represents the vector of applied loads, **F** the vector of internal forces, and $\Delta\mathbf{U}$ the vector of current nodal displacements. In non-linear analysis the load **P** is applied in a series of load steps $\mathbf{P}_{(1)}$, $\mathbf{P}_{(2)}$, $\mathbf{P}_{(3)}$, …

Finite element analysis involves solving the equation above for $\Delta\mathbf{U}$. For the *n*-th load step, the equation is often solved through iterations of the form:

$$\mathbf{K}\Delta\mathbf{U}_{(i+1)} = \mathbf{P}_{(n)} - \mathbf{F}_{(i)} \quad \text{for iterations } i = 0, 1, 2, \ldots$$

The Solution Process, Convergence and Stopping Criterion

The finite element solution process and the definition of convergence are best explained with the simple case of a single force applied to a non-linear spring. In this case, the relationship between the applied load *P* and displacement *U* is:

$$KU = P,$$

where $K(= K(U))$ is the non-linear stiffness of the spring, which is a function of displacement. The non-linear response of the spring to loads is shown in Figure 1.

Let us assume that we already have the solution (displacement $U_{(n)}$) after application of load step $P_{(n)}$ to the spring, and the task now is to determine the spring response – the displacement increment ΔU illustrated in Figure 1– with application of load step $P_{(n+1)}$. Prior to applying the new load step, the resisting (internal) force $F_{(0)}$ in the spring due to its current deformed state is in equilibrium with the applied (external) load $P_{(n)}$.

First, we evaluate the tangent stiffness, $K_{(0)}$, at the origin of the displacement-load curve. Since *RS2* generally uses the initial stiffness method, this stiffness will be used throughout all iterations for the new load step. Next, we calculate the current displacement increment and update the solution:

$$K_{(0)}\Delta U_{(1)} = P_{(n+1)} - F_{(0)}$$

$$\Delta U_{(1)} = K_{(0)}^{-1}(P_{(n+1)} - F_{(0)})$$

$$U_{(n+1)} = U_{(n)} + \Delta U_{(1)}$$

From the current displacement state, we can calculate the internal force, $F_{(1)}$, in the spring (see Figure 1). At this stage, the current force error or load imbalance $P_{(n+1)} - F_{(1)}$ is quite large. From Figure 1, it is evident that a key aim of the iterations is to reduce the load imbalance to zero, or at least a very small number.

For the next iteration, we start at the new, more accurate estimate of the displacement $U_{(n+1)}$ and apply the same load step $P_{(n+1)}$. This time we obtain a displacement increment, $\Delta U_{(2)}$ which is smaller than the increment $\Delta U_{(1)}$ of the previous iteration. Following the procedures described previously, we calculate an updated internal force, $F_{(2)}$, that is closer to equilibrating the applied load.

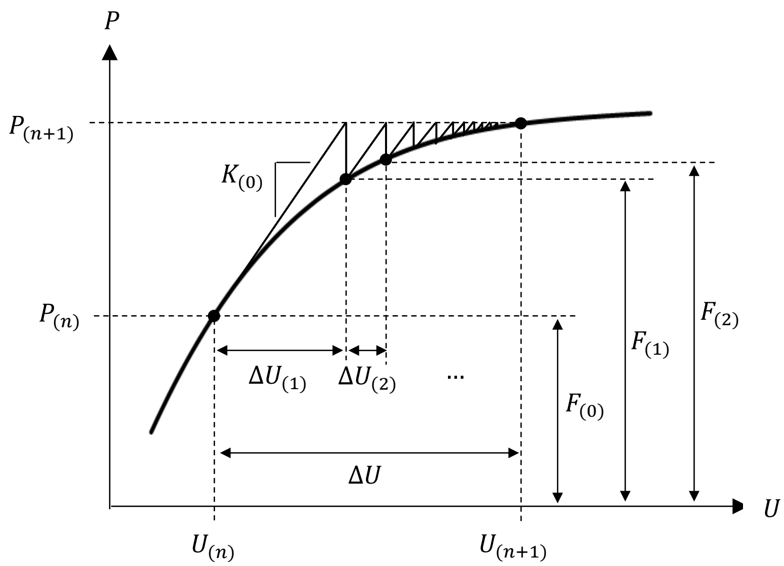

Figure 1: The non-linear response of a spring to applied loads. This problem has a single degree of freedom. The figure also illustrates the iterative finite element procedure for determining the spring's behaviour under applied loads.

Typically, with continued iterations, not only does the load imbalance $P_{(n+1)} - F_{(i)}$ grow smaller and smaller, the displacement increments $\Delta U_{(i)}$ also approach zero, and updates of $U_{(n+1)}$ approach the true solution. In order not to iterate unnecessarily long, we can decide to terminate calculations when the results are "sufficiently close" according to some stopping criterion or criteria.

RS[3] allows you to choose one of the following options for stopping criteria:

- **Absolute Energy**
- **Absolute Force**
- **Absolute Force & Energy**

as defined by the following equations. Although the equations below are written in vector notation, they can be readily applied to our simple, scalar special case.

Absolute Energy Criterion

Energy convergence is satisfied when:

$$\frac{\sum_{j=1}^{N}\left|\left(p_{(n+1),j}-f_{(i),j}\right)\Delta u_{(i),j}\right|}{\sum_{j=1}^{N}\left|\left(p_{(n+1),j}-f_{(0),j}\right)u_{(1),j}\right|} < (specified\ tolerance)$$

where $p_{(n+1),j}$, $f_{(i),j}$, $f_{(0),j}$, $\Delta u_{(i),j}$ and $u_{(1),j}$ are the components (N in total) of $\mathbf{P}_{(n+1)}$, $\mathbf{F}_{(i)}$, $\mathbf{F}_{(0)}$, $\Delta\mathbf{U}_{(i)}$ and $\mathbf{U}_{(1)}$ respectively.

Absolute Force

Force convergence is satisfied when:

$$\frac{\left\|\mathbf{P}_{(n+1)}-\mathbf{F}_{(i)}\right\|}{\left\|\mathbf{P}_{(n+1)}-\mathbf{F}_{(0)}\right\|} < (specified\ tolerance)$$

Absolute Force & Energy

The program will check both conditions at the same time, force and energy.

Our experience

In our experience, for the range of problems solved with *RS3*, the Absolute Force & Energy criteria have been the most consistent and reliable.