



RS2

Thermal Analysis

Theory Manual

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1. Introduction

1.1. Overview

RS2 can be used to model various applications involving thermal analysis such as ground stabilization using artificial ground freezing technique, thermal design for geostructures using insulation to prevent soil freezing or using thermosyphon to prevent soil thawing, effects of seasonal changes on the regional water flow, effects of temperature changes to the geotechnical design due to thermal expansion/contraction. Various types of boundary conditions were implemented into the program to accommodate users in such applications. Users can assign temperature, flux, convective surfaces and thermosyphons. Climate input was also accommodated using a special type of boundary condition call surface energy balance where it accounts for air temperature, wind speed, snow depth, solar radiation, earth emissions and evaporation rate. More details on example applications can be found at the [RS2 Thermal Example Manual](#), and the model files and pdf can be downloaded from [here](#).

RS2 also support thermal-hydro-mechanical coupling analysis where all 3 phases were considered in the simulation by including pore-ice blockage, force convection, natural convection as well as thermal expansion behaviors. The program has gone through a vigorous testing and verifying against the theoretical solutions for the features. The verification documents can be found at the [RS2 Thermal Verification Manual](#), and the model files can be downloaded from [here](#). Details of the theory and formulation of the thermal module in RS2 can be found below.

1.2. Governing equations

1.2.1. Steady

Steady state thermal flow is the condition that the temperature does not depend on time. In RS2, the finite element method used for steady state thermal flow is by linear triangular elements.

For a two-dimensional heat transfer case the equation can be represented by (Lewis et al, 2004),

$$[K]\{T\} = \{f\} \quad (1.1)$$

where

$$[K] = \int_{\Omega} [B]^T [D] [B] d\Omega + \int_{\Gamma} h [N]^T [N] d\Gamma \quad (1.2)$$

$$\{f\} = \int_{\Omega} G [N]^T d\Omega - \int_{\Gamma} q [N]^T d\Gamma + \int_{\Gamma} h T_{\infty} [N]^T d\Gamma \quad (1.3)$$

In above equations, $\{T\}$ is the temperature distribution vector, $[K]$ is the global stiffness matrix, $\{f\}$ is the forcing vector, which represents the heat flux boundary condition.

In linear triangulation elements, the temperature gradient matrix can be written as

$$\{g\} = \begin{Bmatrix} \frac{dT}{dx} \\ \frac{dT}{dy} \end{Bmatrix} = \begin{bmatrix} \frac{dN_i}{dx} & \frac{dN_j}{dx} & \frac{dN_k}{dx} \\ \frac{dN_i}{dy} & \frac{dN_j}{dy} & \frac{dN_k}{dy} \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} = [B]\{T\} \quad (1.4)$$

So $[B]$ is provided as

$$[B] = \begin{bmatrix} \frac{dN_i}{dx} & \frac{dN_j}{dx} & \frac{dN_k}{dx} \\ \frac{dN_i}{dy} & \frac{dN_j}{dy} & \frac{dN_k}{dy} \end{bmatrix} \quad (1.5)$$

$[N]$ is the shape function matrix, h is the heat transfer coefficient, q is the boundary surface heat flux, Ω is the function domain, Γ is the boundary of the domain, G is the internal heat generation, and T_∞ is the atmospheric temperature.

$[D]$ is the thermal conductivity matrix, and it can be written as

$$[D] = \begin{bmatrix} k_x(T) & 0 \\ 0 & k_y(T) \end{bmatrix} \quad (1.6)$$

Where $k_x(T)$, and $k_y(T)$ are the temperature-dependent thermal conductivities in the x and y directions respectively.

1.2.2. Transient

Transient heat flow is the condition that temperature varies with respect to time. The Galerkin method is used to develop transient heat flow equations. The numerical solution for the temperature distribution can be written as (Lewis et al, 2004),

$$[C] \left\{ \frac{\partial T}{\partial t} \right\} + [K]\{T\} = \{f\} \quad (1.7)$$

Or

$$[C_{ij}] \left\{ \frac{\partial T_j}{\partial t} \right\} + [K_{ij}] \{T_j\} = \{f_i\} \quad (1.8)$$

For each node,

$$[C_{ij}] = \int_{\Omega} \rho c_p N_i N_j d\Omega \quad (1.9)$$

$$[K_{ij}] = \int_{\Omega} \left[k_x(T) \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \{T_j\} + k_y(T) \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \{T_j\} \right] d\Omega + \int_{\Gamma} h N_i N_j d\Gamma \quad (1.10)$$

And

$$\{f_i\} = \int_{\Omega} N_i G d\Omega - \int_{\Gamma_q} q N_i d\Gamma_q + \int_{\Gamma_q} N_i h T_a d\Gamma \quad (1.11)$$

In matrix form,

$$[C] = \int_{\Omega} \rho c_p [N]^T [N] d\Omega \quad (1.12)$$

$$[K] = \int_{\Omega} [B]^T [D] [B] d\Omega + \int_{\Gamma} h [N]^T [N] d\Gamma \quad (1.13)$$

And

$$\{f\} = \int_{\Omega} G [N]^T d\Omega - \int_{\Gamma_q} q [N]^T d\Gamma_q + \int_{\Gamma_q} h T_a [N]^T d\Gamma \quad (1.14)$$

In above equations, T is the temperature, t is the time, $\{T\}$ and $\{T_j\}$ are the temperature distribution vectors, $\{f\}$ and $\{f_i\}$ are the forcing vectors, ρ is the density, c_p is the specific heat capacity, N_i and N_j are the shape functions, $[N]$ is the shape function matrix, h is the heat transfer coefficient, G is the internal heat generation, q is the boundary heat flux, T_a is the atmospheric temperature, and i and j represent the nodes.

Over the domain Ω , the boundary is given by $\Gamma_b \cup \Gamma_q = \Gamma$ and $\Gamma_b \cap \Gamma_q = 0$, where the whole boundary (Γ) is consisted of a prescribed temperature boundary (Γ_b) and a prescribed heat flux boundary (Γ_q).

$[B]$ is the gradient matrix of the shape function (see eqn (1.5)) and $[D]$ is the thermal conductivity matrix,

$$[D] = \begin{bmatrix} k_x(T) & 0 \\ 0 & k_y(T) \end{bmatrix} \quad (1.15)$$

Where $k_x(T)$, and $k_y(T)$ are the temperature-dependent thermal conductivities in the x and y directions respectively.

It is noted that for the transient state, if latent heat is included, the energy for phase change is considered. Compared to without latent heat (eqn (1.9) or (1.12)), the matrix $[C]$ in eqn (1.7) will become,

$$[C] = \int_{\Omega} \rho c_p [N]^T [N] d\Omega + \int_{\Omega} \rho_w L_w m_2^i [N]^T [N] d\Omega \quad (1.16)$$

Where ρ_w is the water density, L_w is the latent water, and m_2^i is the slope of unfrozen water content curve.

1.2.3. Time Step

To solve time-dependent differential equations in transient heat flow problems, the direct time integration algorithm is used, where a factor of the time step (Δt) is involved. Due to stability considerations, between the two numerical approximation methods, the implicit scheme is chosen over the explicit scheme.

For the explicit scheme, the error amplifies unacceptably when the time step size exceeds a limit value, saying that the scheme is conditionally stable. The implicit scheme is unconditionally stable on the other hand, where the stability is regardless of the time step size.

In finite element analysis, smaller time step sizes can increase result accuracy. However, for heat transfer analysis, the time step size (Δt) should not be less than a limit value. The minimum time step size is in relation to the element size, given as (Bergheau and Fortunier, 2008),

$$\Delta t \geq \frac{\rho c_p e^2}{6k\nu} \quad (1.17)$$

Where ρ is the density, c_p is the specific heat capacity, e is the typical element size (e.g. the length of a side of the element), and k is the thermal conductivity. The index ν is with reference to the time integration algorithm. $\nu = 0$ corresponds to an explicit scheme, $\nu = 1$ corresponds to an implicit scheme, and $0 < \nu < 1$ refers to a semi-implicit scheme. Since the implicit scheme is chosen as discussed above, $\nu = 1$ is applied to the equation.

If the time step size Δt is smaller than the limit value, the space oscillations of the solution can appear during the initial moments of the simulation. Alternatively, to satisfy equation (1.17), a denser mesh can be applied to correspondent locations, so as to lower the element size e , and thus lowers the minimum time step size.

1.3. Heat Convection

The heat transfer mechanism due to a fluid motion is called heat convection. When the fluid motion is caused by hydraulic head differences, the convection process is called a forced convection. When the fluid motion occurs due to the density variations induced by the temperature differences, such convection process (buoyancy-driven motion) is called a free or natural convection.

1.3.1. Forced Convection

For the forced convection, governed by the momentum and continuity equations, the fluid flow in 2D is given as (Frivik and Comini, 1982),

$$u = -k_p \frac{\partial p}{\partial x} \quad (1.18)$$

$$v = -k_p \frac{\partial p}{\partial y} \quad (1.19)$$

Where

$$p = p' + \rho g z \quad (1.20)$$

In above equations, u and v are the flow velocity components in the x,y directions, k_p is the coefficient of permeability, p is the driving pressure, p' is the total fluid pressure, g is the acceleration of gravity, and z is the vertical distance from a horizontal plane, taken as positive in the upwards direction.

The general matrix equation for steady state can be found in eqn (1.1), and that for transient state is shown in eqn (1.7). However, compared to heat conductivity, an additional fluid flow component is accounted for in the K matrix, given as

$$K_{ij} = \int_{\Omega} \left[k_x(T) \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \{T_j\} + k_y(T) \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \{T_j\} \right] d\Omega + \int_{\Omega} c_l N_i \left(u \frac{\partial N_j}{\partial x} + v \frac{\partial N_j}{\partial y} \right) d\Omega + \int_{\Gamma} h N_i N_j d\Gamma \quad (1.21)$$

Where c is the heat capacity per unit volume, l is the liquid, u and v represent the fluid velocity components in the x, y directions.

The Peclet Number

The Peclet number (Pe) is often employed in forced convection heat transfer problems. If the Peclet number exceed 1000, the algorithm will become unstable, where spurious temperature increases. The same problem also applies to the situation when the fluid flows along a boundary that has rapid changes in temperature. To reduce the spurious temperature, special treatment may be needed (i.e. Unwind treatment). However, for most soil problem, this is not an issue because fluid flow is small. If the issue arises, the special treatments will need to be implemented.

The Peclet number can be calculated by (Lewis et al, 2004),

$$Pe = RePr = \frac{u_a L}{\alpha} \quad (1.22)$$

Where Re is the Reynolds number defined as eqn (1.23) below, Pr is the Prandtl number defined as eqn (1.24) below, α is the thermal diffusivity calculated as eqn (1.25) below, u_a is the average inlet velocity, L is the characteristic dimension, ν is the kinematic viscosity calculated by viscosity over density of the fluid (μ/ρ), k is the thermal conductivity, and c_p is the specific heat capacity.

$$Re = \frac{u_a L}{\nu} \quad (1.23)$$

$$Pr = \frac{\nu}{\alpha} \quad (1.24)$$

$$\alpha = \frac{k}{\rho c_p} \quad (1.25)$$

1.3.2. Natural Convection

In natural convection, the temperature difference induces a density-dependent transport, where the fluid flow is driven by buoyancy force. To account for the effect of buoyancy force, the body forces are adjusted based on the temperature. Governed by the Darcy's equation, the fluid flow equation is given as (Frind, 1982),

$$q_i = -K_{wij} \left(\frac{\partial \psi}{\partial x_j} + \rho_r \eta_j \right) \quad (1.26)$$

q_i is the flux in the i direction, K_{wij} is the permeability, ψ is the equivalent freshwater head, ρ_r is the relative density, and $\eta_j = 1$ indicates the vertical direction while $\eta_j = 0$ indicates the horizontal directions.

where

$$K_{w_{ij}} = k_{w_{ij}} \frac{\rho_0 g}{\mu} \quad (1.27)$$

$$\psi = \frac{p}{\rho_0 g} + z \quad (1.28)$$

and

$$\rho_r = \frac{\rho}{\rho_0} - 1 \quad (1.29)$$

The permeability ($K_{w_{ij}}$) equation is provided in eqn (1.27), where $k_{w_{ij}}$ is the permeability tensor, ρ_0 is the reference (freshwater) density, g is the acceleration due to gravity, μ is the dynamic viscosity. The equivalent pressure freshwater head (ψ) equation is provided in eqn (1.28), where p is the pressure, and z is the elevation above datum. The relative density (ρ_r) equation is provided in eqn (1.29).

To perform the finite element analysis, the general matrix equation for transient state is given as below. The hydraulic head changes with respect to time.

$$[H]\{\psi\} + [S]\left\{\frac{\partial\psi}{\partial t}\right\} + \{G\} - \{F\} = 0 \quad (1.30)$$

Where $[H]$ is the permeability matrix, $[S]$ is the fluid mass matrix, $\{G\}$ is the body force vector, and $\{F\}$ is the boundary flux vector. The matrices are shown below.

$$[H_{ij}] = \int_{\Omega} \left[k_{w_x}(\psi) \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + k_{w_y}(\psi) \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] d\Omega \quad (1.31)$$

Where ψ is the total head, k_{w_x} and k_{w_y} are the permeability in x and y directions, N_i and N_j are the shape functions.

$$[S_{ij}] = \int_{\Omega} S_s N_i N_j d\Omega \quad (1.32)$$

Where S_s is the specific storage defined as eqn (1.33) below, N_i and N_j are the element shape functions. In eqn (1.33), α is the compressibility of the porous medium, β is the compressibility of the fluid, and θ is the porosity.

$$S_s = \rho_0 g (\alpha + \theta \beta) \quad (1.33)$$

$$\{G_i\} = \int_{\Omega} k_{wy}(\psi) \gamma \bar{c} \frac{\partial N_i}{\partial y} d\Omega \quad (1.34)$$

Where \bar{c} is the average concentration over an element; γ is a constant derived from the maximum density of fluid (ρ_{max}) and the reference (freshwater) density (ρ_0), given as,

$$\gamma = \frac{\rho_{max}}{\rho_0} - 1 \quad (1.35)$$

$$\{F_i\} = \int_{\Gamma} q_n N_i d\Gamma \quad (1.36)$$

Where q_n is the heat flux in the direction normal to the boundary.

1.4. Pore-ice Induced Seepage

Pore-ice induced seepage occurs when ice forming in the pores of soils and rocks. In such cases, the hydraulic models that account for unsaturated behaviors (i.e. Van Genuchten) will be affected. To incorporate the pore-ice induced seepage effect, the additional suction on top of the existing pore pressure needs to be calculated. Based on Clapeyron equation of equilibrium relationship between the temperature and pressure in freezing soil, the additional pore-ice pressure can be calculated with the expression (Kurylyk and Watanabe, 2013),

$$\frac{dP_i}{dT} = \frac{H_f \rho_i}{(T + 273.15)} \quad (1.37)$$

Where P_i is the equilibrium gauge pressures in partially frozen soil for the ice phases, T is the equilibrium freezing temperature in °C, H_f is the latent heat of fusion, and ρ_i is the ice density.

Note: Only hydraulic models that accounted for unsaturated behaviors will be affected by pore-ice blockage.

1.5. Thermal Dispersivity

In heat transfer problems, the degree of thermal dispersion will affect the thermal conductivity. To evaluate the impact, two parameters need to be defined: the longitudinal dispersivity (λ_{lo}) and transverse dispersivity (λ_{tr}).

With thermal dispersivity parameters, the nodal matrix for the thermal dispersion tensor ($D_{eff_{ij}}$) is provided as eqn (1.38) below in finite element heat analysis, where u and v are inlet velocities in x and y directions.

$$[D_{eff_{ij}}] = \frac{1}{|\mathbf{u}|} \begin{bmatrix} \lambda_{lo}u^2 + \lambda_{tr}v^2 & (\lambda_{lo} - \lambda_{tr})uv \\ (\lambda_{lo} - \lambda_{tr})uv & \lambda_{tr}u^2 + \lambda_{lo}v^2 \end{bmatrix} \quad (1.38)$$

For both steady and transient states (see eqn (1.1) and (1.7)), an additional term will be included in the \mathbf{K} matrix to account for the thermal dispersion tensor $[\mathbf{D}_{eff}]$, as given in eqn (1.39) below. For comparison, the \mathbf{K} matrices without the impact of thermal dispersivity are provided in eqn (1.2) or (1.13).

$$[\mathbf{K}] = \int_{\Omega} [\mathbf{B}]^T [\mathbf{D}] [\mathbf{B}] d\Omega + \int_{\Gamma} h [\mathbf{N}]^T [\mathbf{N}] d\Gamma + \int_{\Omega} [\mathbf{B}]^T [\mathbf{D}_{eff}] [\mathbf{B}] d\Omega \quad (1.39)$$

1.6. Couple Thermal Strain

When the thermal effect is considered in strain analysis, an additional thermal strain ($\Delta\varepsilon_{ii}^T$) is accounted for. The change of stress is generally calculated based on strain increments using Hooke's law (eqn (1.40)),

$$\Delta\sigma_{ij} = D^e \Delta\varepsilon_{ij} \quad (1.40)$$

Where $\Delta\sigma_{ij}$ is the incremental stress tensor, D^e is the elastic constitutive tensor, and $\Delta\varepsilon_{ij}$ is the incremental strain tensor.

Under thermal exposure, the thermal strain is calculated by,

$$\Delta\varepsilon_{ii}^T = \alpha_T \Delta T \quad (1.41)$$

Where $\Delta\varepsilon_{ii}^T$ is the thermal strain ($i = 1$ or 3 , i.e. normal strain components), α_T is the coefficient of linear thermal expansion, ΔT is the change in temperature.

Adding the thermal strain (eqn (1.41)) to eqn (1.40), the change of stress becomes,

$$\Delta\sigma_{ij} = D^e (\Delta\varepsilon_{ij} - \Delta\varepsilon_{ii}^T) \quad (1.42)$$

2. Boundary Conditions

2.1. Regular Boundary Conditions

For a thermal model, regular boundary conditions include temperature boundary condition, flux boundary condition, and nodal flux boundary condition.

2.1.1. Temperature boundary condition

For the temperature boundary condition, the temperature can be applied as constant or time dependent function. Several methods for constant temperature are provided, including uniform, vertical incremental, horizontal incremental, and initial and end values. If the temperature is time dependent, a temperature-time function will need to be defined.

It is optional to include the N factor in time dependent temperature boundary condition. The N factor is an empirical coefficient to consider the air temperature impact on the ground surface temperature. With inputs of n-factor for thawing (N_{thaw}), n-factor for freezing (N_{freeze}), and the freezing temperature (T_f), the temperature will be calculated as,

$$T = \begin{cases} N_{thaw}(T - T_f) + T_f & \text{for } T > T_f \\ N_{freeze}(T - T_f) + T_f & \text{for } T \leq T_f \end{cases} \quad (2.1)$$

2.1.2. Flux and nodal flux boundary condition

The flux and nodal flux boundary define a prescribed heat flux out of the model (as a unit per length per unit of width) over time. It is mentioned that for time dependent flux boundary condition, the climate data can be defined and used as the time function. See section 2.6.3 for details.

2.2. Convection Boundary Condition

A convection boundary condition applies to surfaces exposed to convective heat transfer. A convective boundary defines an external environment of a medium (denoted as a fluid) with a certain temperature, where this temperature is transferred to boundary due to natural/forced convection.

For example, the air temperature may be different from the ground surface temperature, a convection boundary can be used in this case. An additional heat transfer coefficient is required for the temperature of the medium. For both medium temperature and heat transfer along the boundary, the apply method can be either constant or time dependent. If the convection boundary condition is applied to a node, the perimeter will be required to inputted. If the user select a working temperature range then the convection boundary condition will only work when the medium temperature falls in that range.

2.3. Thermosyphon Boundary Condition

A thermosyphon system can be used in arctic area to keep soils stay frozen. A thermosyphon boundary condition is similar to convection boundary except for its two limitations. The thermosyphon will not work when: a) the air temperature is higher than a limit, and b) the difference between the air temperature and ground is smaller than the threshold value ΔT . In RS2, the air temperature, delta temperature (ΔT), and the max temperature need to be inputted. The heat transfer rate at a thermosyphon boundary can be determined with three methods: manual, arctic foundation, or Zarling and Haynes.

i. Manual

For the manual method, the heat transfer rate is determined by a heat transfer coefficient, which can either be inputted as a constant value, or a wind speed dependent function.

ii. Arctic Foundation

Arctic Foundations Inc. (Yarmak, 1983) have suggested an empirical expression to describe the linear relationship between the heat transfer rate and the wind speed, given different units, (Haynes and Zarling, 1988),

$$C = 8.97 + 34.3B + 9.8V_w \quad [W/^\circ C] \quad (2.2)$$

$$C = 17 + 65B + 98.3V_w \quad [Btu/hr \cdot ^\circ F] \quad (2.3)$$

Where C is the thermal conductance, B is a factor based on the exposure of the thermosyphon ($B = 1$ in open area, $B = 0.75$ at an outside corner of a building, $B = 0.5$ next to a building, and $B = 0.25$ at an inside corner of a building), where $B = 0.5$ in RS2, and V_w is the wind speed. The wind speed can be inputted as a constant value or a function of time.

The equivalent thermal conductivity (k_{eq}) accounted for the syphon length will be,

$$k_{eq} = \frac{C}{\text{Syphon length}} \quad (2.4)$$

iii. Zarling and Haynes

Haynes and Zarling (1988) suggested that the evaporator slope angle has an impact on the relationship between thermal conductance and wind speed. The behavior is different depending on the working fluid (ammonia or carbon dioxide) within the thermosyphon. Table 2.1 and Table 2.2 below present the overall heat transfer conductance for ammonia and carbon dioxide thermosyphon respectively, where V_w is the wind speed (m/s).

Table 2.1 Empirical expressions for overall heat transfer conductance CO₂ thermosyphon

Slope of evaporator (degrees)	Conductance (W/°C)	Standard error of estimate
0	$8.70 + 12.29V_w^{0.83}$	1.48
3	$6.10 + 25.85V_w^{0.55}$	3.65
6	$19.47 + 21.13V_w^{0.71}$	5.12
9	$20.79 + 19.11V_w^{0.80}$	4.36
12	$16.50 + 26.00V_w^{0.62}$	2.42

Table 2.2 Empirical expressions for overall heat transfer conductance NH₃ thermosyphon

Slope of Evaporator (degrees)	Conductance (W/°C)	Standard Error of Estimate
0	$23.74 + 8.17V_w^{0.46}$	1.62
3	$26.44 + 11.90V_w^{0.74}$	2.03
6	$25.81 + 11.98V_w^{0.75}$	1.33

2.4. Insulation Boundary Condition

Insulation boundary conditions are applied only to joint elements. It describes the thermal conductance between the joint and the body. Three options are provided: perfect insulate, none, and insulate. Perfect insulate means no heat transfer between surfaces, none means heat can freely pass through, and insulate means heat can transfer at a certain degree defined by an inputted conductivity value.

2.5. Free Boundary Condition

Free boundary condition represents no heat transfer over the boundary.

2.6. Thermal Time Dependent Boundary Condition

For thermal boundary conditions, time functions can be inputted to define correspondent time dependent factors at a certain time. Three types of time function are available in RS2 thermal analysis: tabular, sinusoidal, and climate. The tabular and sinusoidal methods are applicable to all thermal boundary conditions, and the climate method is only applicable to the flux boundary condition.

2.6.1. Tabular

For the tabular method, a table of time vs. value should be inputted.

2.6.2. Sinusoidal

For the sinusoidal time function, a sinusoidal graph will be defined based on the amplitude, period, phase, minimum steps, minimum value, and the maximum value.

2.6.3. Climate

The climate data are used to estimate the temperature for the ground surface. According to the ground surface energy balance, the total net heat flux at the ground-air interface can be expressed as (Hwang, 1976),

$$Q_G = QL - QS + QC + QE \quad (2.5)$$

Where Q_G is the heat flux transferred through the ground, QL is the net long wave radiation leaving the surface, QS is the net short wave or net solar radiation absorbed at the surface, QC is the net convective heat flux, and QE is the heat flux from evaporation (note that only evaporation is considered, rather than evapotranspiration).

The net radiation including long and short waves ($QL - QS$) can either be user defined or calculated from solar radiation. If the net radiation is user defined, a table of net radiation flux vs time need to be inputted. If the net radiation is calculated from solar radiation, the net long wave (QL) and net solar radiation (QS) radiation will be estimated separately using equations as described below.

Net Long Wave Radiation

The net long wave radiation leaving the surface (QL) can be calculated as (Hwang, 1976),

$$QL = E\sigma T_{sur}^4(1 - G) \quad (2.6)$$

Where E is the surface emissivity, RS2 uses $E = 0.95$, σ is the Stefan-Boltzmann radiation constant ($= 1.713 \times 10^{-1} \frac{BTU}{h \cdot ft^2 \cdot ^\circ R^4}$), and T_s is the surface temperature ($^\circ R$), and G is the greenhouse factor, $G = 1$ is used in RS2.

Net Solar (Shortwave) Radiation

The net solar radiation absorbed at the surface (QS) can be estimated in two ways: user defined, or automatically calculated.

User Defined

When solar radiation data are defined by the user, the net solar radiation (QS) will be (Hwang, 1976),

$$QS = SW \times (1 - a) \times VF \quad (2.7)$$

Where SW is the solar radiation flux measured on a horizontal surface, available at meteorological stations, which will need to be inputted by the user, a is the albedo, which is a measure of reflected solar radiation as a percentage of the incident radiation, and VF is the view factor which includes both shadow effects from obstruction and inclination effects due to non-horizontal surfaces, $VF = 1$ is used in RS2.

Automatically Calculated

When the net solar radiation is automatically calculated without the solar radiation data defined, two additional inputs are required instead: daylight and latitude. The daylight input (shown as the variable n in equation (2.15)) is the actual duration of sunlight during the day, it can either be selected as “clear all day” to become the maximum possible during of sunlight ($n = N$ in equation (2.12)), or defined as an input daylight function.

The extraterrestrial radiation will be calculated first as (Allen et al., 1998),

$$q_{ext} = \frac{(24)(60)}{\pi} G_{sc} d_r [\omega_s \sin \varphi \sin \delta + \cos \varphi \cos \delta \sin \omega_s] \quad (2.8)$$

Where q_{ext} is the extraterrestrial radiation, G_{sc} is the solar constant (118 MJ/m²/day), d_r is the inverse relation distance from earth to sun, see equation (2.9), ω_s is the sunset hour angle, see equation (2.11), φ is the latitude, and δ is the solar declination, which is the angle between the Sun's ray and the plane of the Earth's Equator, see equation (2.10).

$$d_r = 1 + 0.033 \cos \left(\frac{2\pi}{365} J \right) \quad (2.9)$$

$$\delta = 0.409 \sin \left(\frac{2\pi}{365} J - 1.39 \right) \quad (2.10)$$

$$\omega_s = \cos^{-1}(-\tan \varphi \tan^{-1} \delta) \quad (2.11)$$

In equations (2.9) to (2.11) above, J is the day of the year between 1 (January 1st) and 365 (December 31st) or 366 during a Leap year, which is an input by the user.

Then, the shortwave radiation (q_s) can be calculated as,

$$q_s = \left(a_s + b_s \frac{n}{N} \right) q_{ext} \quad (2.12)$$

Where n is the actual duration of sunlight, N is the maximum possible duration of sunshine or daylight, see equation (2.16), q_{ext} is the extraterrestrial radiation, a_s and b_s are the regression constants.

$$N = \frac{24}{\pi} \omega_s \quad (2.13)$$

Next, the net solar radiation (QS) can be obtained from the shortwave radiation (q_s), similar to equation (2.7) as (Dingman, 2008),

$$QS = q_s \times (1 - a) \times VF \quad (2.14)$$

Net Convective Heat Flux

The net convective heat flux (QC) can be calculated as (Saito and Simunek, 2009),

$$QC = C_a \frac{(T_s - T_a)}{r_a} \quad (2.15)$$

Where r_a is the aerodynamic resistance to water vapor or heat flow from a soil surface to the atmosphere, and C_a is the volumetric heat capacity of air. r_a can be calculated by,

$$r_a = \frac{1}{uk^2} \left[\ln \left(\frac{z_{ref} - d + z_h}{z_h} \right) + \Psi_h \right] \times \left[\ln \left(\frac{z_{ref} - d + z_m}{z_m} \right) + \Psi_m \right] \quad (2.16)$$

Where u is the wind speed, k is the von Karman's constant (=0.41), z_{ref} is the reference height for measurements (typically 1.5 m), d is the zero-plane displacement height, $z_{h,m}$ are the surface roughness height for heat and momentum flux respectively, and $\Psi_{h,m}$ are the atmospheric stability correction factor for momentum and heat flux respectively.

The values of d , z_h , and z_m depend on the value of vegetation height. d is assumed to be 2/3 of the vegetation height, z_h is assumed to be 0.1 times the vegetation height, and z_m is assumed to be 0.123 times the vegetation height. The atmospheric stability correction factors ($\Psi_{h,m}$) are assumed to be equal. Their values are determined by the Richardson number, R_i (Saito and Simunek, 2009),

$$R_i = \frac{g(z_{ref})(T_a - T_g)}{(T_a)u^2} \quad (2.17)$$

$$\Psi = (1 - 10R_i)^{-1}, \text{ for } R_i < 0 \quad (2.18)$$

$$\Psi = (1 + 10R_i)^{-1}, \text{ for } R_i > 0 \quad (2.19)$$

It is noted that whether to account for the effect of vegetation depth is optional in RS2. When the vegetation depth is considered, the vegetation height data are required to be inputted by the user. When the vegetation depth effect is neglected, a value of 0 for vegetation height is used to calculate d , z_h , and z_m .

Evaporation Heat Flux

The evaporation heat flux (QE) can be obtained from the actual evaporation and the latent heat of air, given as,

$$QE = AE \times L_a \quad (2.20)$$

Where AE is the actual evaporation, which should be inputted as a table of evaporation vs. time, and L_a is the latent heat of air ($L_a = 2.45e3 \frac{MJ}{kg}$, assuming water density is $1000 \frac{kg}{m^3}$).

Snowpack Effect

It is optional to include the snowpack effect in ground surface energy balance. When snowpack is included, the snow depth data are required to be inputted by the user. The heat flux through the snow can be expressed as,

$$Q_{snow} = -k_{snow} \frac{(T_g - T_{snow})}{h_{snow}} \quad (2.21)$$

And the ground surface energy balance equation will become,

$$Q_G = QL - QS + QC + QE + Q_{snow} \quad (2.22)$$

3. Thermal Material

3.1. Element Types

The thermal properties are different in liner and solid elements.

3.1.1. Liner

For liners in steady states, the thermal conductivity and thermal expansion need to be inputted; for that in transient states, the thermal conductivity, heat capacity, and thermal expansion coefficient (optional) need to be defined. It is noted that the defined thermal conductivity and specific heat capacity for liners are constant values.

3.1.2. Solid

For solid elements in steady states, the thermal conductivity, thermal soil unfrozen water content, thermal expansion (optional), and dispersivity (optional) need to be defined. For solid elements in transient states, the heat capacity needs to be specified in addition. The available methods to define each parameter are explained in the following section.

3.2. Thermal Conductivity

Thermal conductivity is a crucial thermal property evaluating the degree to which the specified material conducts heat. There are seven methods available to determine the thermal conductivity in RS2.

3.2.1. Constant

The thermal conductivity can be inputted as constants. For soils, both unfrozen conductivity and frozen conductivity need to be defined; in addition, the frozen temperature needs to be specified as well. The thermal conductivities for a number of common construction materials are listed in Table 3.1 below.

Table 3.1 Thermal Conductivity Values for Common Construction Materials (U.S. Army Corps of Engineers.)

Type of Material	Description	Unit weight (lb/ft ³)	Conductivity (Btu/ft·hr·°F)	Conductivity (W/m-K)
Asphalt paving mixture	Mix with 6% by weight cutback asphalt	138	0.86	1.49
	with sand and gravel or stone aggregate (oven-dried)	140	0.75	1.3
Concrete	with sand and gravel or stone aggregate (not dried)	140	1	1.73
	-	120	0.43	0.744

	with lightweight aggregates, including	expanded shale or clay	100	0.3	0.519
		expanded slate	80	0.21	0.363
		expanded slags; cinders	60	0.14	0.242
		pumice; perlite	40	0.096	0.166
		vermiculite	30	0.075	0.13
		cellular concretes	20	0.058	0.1
Wood	Maple, oak and similar hardwoods		45	0.092	0.159
	Fir, pine and similar softwoods		32	0.067	0.116
Building boards	Asbestocement board		120	0.33	0.571
	Plywood		34	0.067	0.116
	wood fiberboards, laminated or homogeneous		26,33	0.035, 0.046	0.0605, 0.0796
	Wood fiberhardboard type		65	0.12	0.208
Blanket and batt insulation	Mineral wool, fibrous form, processed from rock, slag, or glass		1.5-4.0	0.022	0.0381
	Wood fiber		3.2-3.6	0.021	0.0363
Board and slab insulation	Cellular glass		9	0.032	0.0553
	Corkboard (without added binder)		6.5-8.0	0.022	0.0381
	Glass fiber		9.5-11.0	0.021	0.0363
	Wood or cane fiberinterior finish (plank, tile, lath)		15	0.029	0.0502
	Expanded polystyrene		1.6	0.024	0.0415
	Expanded ureaformaldehyde		1	--	--
	Expanded perlite		9.5-11.5	--	--
	Polyurethane foam		1.5-3.0	--	--
	Mineral wool with resin binder		15	0.023	0.0398
	Mineral wool with asphalt binder		15	0.026	0.045
Loose fill insulation	Cork, granulated		5-12	--	--
	Expanded perlite		3-4	--	--
	Mineral wool (glass, slag, or rock)		2-5	0.025	0.0432
	Sawdust or shavings		8-15	0.037	0.064
	Straw		7-8	--	--
	Vermiculite (expanded)		7.0-8.2	0.04	0.0692
	Wood fiber: redwood, hemlock, or fir		2.0-3.5	0.025	0.0432
Miscellaneous	Aluminum		168	118	204
	Copper		549	220	381
	Ductile iron		468	30	51.9
	Glass fiber		164	0.46	0.796
	Ice		57	1.28	2.21
	Snow, new, loose		5.3	0.05	0.0865
	Snow on ground		18.7	0.13	0.225
	Sno, difted and compacted		31.2	0.4	0.692
	Steel		487	25.8	44.6
	Water, average		62.4	0.35	0.605

3.2.2. Johansen

The Johansen approach (Johansen, 1977) (Farouki, 1981) uses soil state, types, fraction of soil components, and thermal conductivity of water, ice, and solid components to calculate the thermal conductivity.

The suggested quartz content values for common soils are listed in Table 3.2 below. The use of quartz content in equations are discussed in this section.

Table 3.2 **Quartz content of common soils (Tarnawski et al., 2009)**

Chinese Soil (Lu et al., 2007)		Kerstern's soil (Tarnawski et al., 2009)	
Soil name	Quartz content (%)	Soil name	Quartz content (%)
Sand 1	75	Lowell sand	77
Sand 2	53	Northway fine sand	23
Sandy loam 3	65	Northway sand	5
Loam 4	54	Fairbanks sand	84
Loam 11	56	Dakota sandy loam	73
Silt loam 5	49	Ramsey sandy loam	53
Silt loam 6	60	Northway silt loam	2
Silty clay loam 7	34	Fairbanks silt loam	40
Silty clay loam 8	33	Fairbanks silty clay loam	28
Clay loam 9	39	Healy clay	31

a. Fine and Coarse Soils and Crushed Rocks

For fine and coarse soils, or crushed rocks, the formula is,

$$k = (k_{sat} - k_{dry})k_e + k_{dry} \quad (3.1)$$

Where k is the soil thermal conductivity, k_{sat} is the saturated conductivity (see eqn 3.4-3.5), k_{dry} is the thermal conductivity of dry nature soil or fry crushed rock material (see eqn 3.2-3.3), and k_e is the Kerstern number (normalized thermal conductivity), which is related to saturation (see eqn 3.8-3.10).

$$k_{dry} = \frac{0.135k_d + 64.7}{2700 - 0.947k_d} \pm 20\%, \text{ for dry nature soils} \quad (3.2)$$

$$k_{dry} = 0.039n^{-2.2}, \text{ for dry crushed rock materials} \quad (3.3)$$

$$k_{sat} = (k_s)^{1-n}(k_w)^n, \text{ for saturated unfrozen soil} \quad (3.4)$$

$$k_{sat} = (k_s)^{1-n}(k_i)^{n-w_u}(k_w)^{w_u}, \text{ for saturated frozen soil} \quad (3.5)$$

$$k_s = 7.7^q(2.0)^{1-q}, \text{ if } q > 0.20 \quad (3.6)$$

$$k_s = 7.7^q(3.0)^{1-q}, \text{ if } q \leq 0.20 \quad (3.7)$$

$$k_e = 0.7 \log(S_r) + 1.0, \text{ for unfrozen coarse grained soil} \quad (3.8)$$

$$k_e = \log(S_r) + 1.0, \text{ for unfrozen fine grained soil} \quad (3.9)$$

$$k_e = S_r, \text{ for frozen soil} \quad (3.10)$$

In above eqns (3.2) to (3.10), k_d is the dry density of soil, n is the soil porosity, k_s is the thermal conductivity of soil particles, q is the quartz content, k_w is the thermal conductivity of water, k_i is the thermal conductivity of ice, w_u the volumetric unfrozen water content, S_r the soil saturation.

b. Peat Soils

For peat soils, the thermal conductivity equations are different under unfrozen and frozen conditions, equations are given in eqn 3.11 and 3.12 below. Three additional parameters need to be inputted: the thermal conductivity of dry peat (k_{dry}), thermal conductivity of saturated unfrozen peat ($k_{sat(u)}$), and thermal conductivity of saturated frozen peat ($k_{sat(f)}$).

$$\sqrt{k} - \sqrt{k_{dry}} = S_r(\sqrt{k_{sat(u)}} - \sqrt{k_{dry}}), \text{ for unfrozen peat} \quad (3.11)$$

$$k = k_{dry} \left(\frac{k_{sat(f)}}{k_{dry}} \right)^{S_r}, \text{ for frozen peat} \quad (3.12)$$

3.2.3. Johansen-Lu

The Johansen-Lu approach is improved based on the Johansen approach. To account for the main limitation in the Johansen approach that, the Kerstern number (k_e) will be negative for situations at low saturations for soils, the relationship between the Kerstern number and water saturation was proposed in Johansen-Lu as (Lu et al, 1977),

$$k_e = \exp\{\alpha[1 - S_r^{(\alpha-\beta)}]\} \quad (3.13)$$

Where α is the parameter related to soil texture. $\alpha = 0.96$ for the coarse soil, and $\alpha = 0.27$ for the fine soil; β is the shape parameter, $\beta = 1.33$; and S_r is the degree of water saturation.

The dry thermal conductivity for soil (k_{dry}) can be estimated in a linear function with soil porosity (n),

$$k_{dry} = -0.56n + 0.51 \quad (3.14)$$

The rest of parameters and then the thermal conductivity will be calculated using the Johansen approach as abovementioned.

3.2.4. De Vries

The De Vries method estimated the thermal conductivity of soil based on the weighted average of the thermal conductivities of their constituents. The formulas are given as (Farouki, 1981),

$$k_{unfrozen} = \frac{\theta_w k_w + F_a(n - \theta_w)k_a + F_s(1 - n)k_s}{\theta_w + F_a(n - \theta_w) + F_s(1 - n)} \quad (3.15)$$

$$k_{frozen} = \frac{\theta_i k_i + \theta_u k_w + F_a(n - \theta_w)k_a + F_s(1 - n)k_s}{\theta_w + F_a(n - \theta_w) + F_s(1 - n)} \quad (3.16)$$

$$F_a = \frac{1}{3} \left\{ \frac{2}{1 + g_a \left(\frac{k_a}{k_w} - 1 \right)} + \frac{1}{1 + g_c \left(\frac{k_a}{k_w} - 1 \right)} \right\} \quad (3.17)$$

$$F_s = \frac{1}{3} \left\{ \frac{2}{1 + 0.125 \left(\frac{k_s}{k_w} - 1 \right)} + \frac{1}{1 + 0.75 \left(\frac{k_s}{k_w} - 1 \right)} \right\} \quad (3.18)$$

$$g_a = 0.035 + 0.298S_r \quad (3.19)$$

$$g_c = 1 - 2g_a \quad (3.20)$$

Where $k_{unfrozen}$ is the thermal conductivity of unfrozen soil, k_{frozen} is the thermal conductivity of frozen soil, k_w is the thermal conductivity of water, k_a is the thermal conductivity of air, k_s is the thermal conductivity of solid particles, k_i is the thermal conductivity of ice, θ_w is the volumetric water content, θ_i is the volumetric ice content, θ_u is the volumetric unfrozen water content, F_a and F_s are shape factors of the air and soil solids, g_a and g_c are the fitting parameters, and S_r is the degree of water saturation.

3.2.5. Cote and Konrad

The Cote and Konrad method uses the relationship between the normalized thermal conductivity (i.e. the Kerstern number) and the degree of saturation to estimate thermal conductivities, which are measured from a large scale of experimental samples.

The normalized thermal conductivity is given as (Cote and Konrad, 2005),

$$k_e = \frac{\kappa S_r}{1 + (\kappa - 1)S_r} \quad (3.21)$$

Where κ (kappa) is an empirical parameter accounting for soil types. The typical values for κ are given in Table 3.3 below.

Table 3.3 Typical Values for κ (Cote and Konrad, 2005)

Material type	κ (Kappa)	
	unfrozen	frozen
Gravels and coarse sands	4.60	1.70
Medium and fine sands	3.55	0.95
Silty and clayey soils	1.90	0.85
Organic fibrous soils (peat)	1.60	0.25

The thermal conductivity for dry soils (k_{dry}) is estimated by an exponential relationship as,

$$k_{dry} = \chi \times 10^{-\eta n} \quad (3.22)$$

Where χ is an empirical value accounting for soil types (W/m-°C), η is an empirical constant, and n is the soil porosity. The typical values for χ and η are provided in Table 3.4 below.

Table 3.4 Typical Values for χ and η (Cote and Konrad, 2005)

Material type	χ (chi) (W/m-°C)	η (eta)
Crushed rock and gravels	1.70	1.80
Natural mineral soils	0.75	1.20
Organic fibrous soils (peat)	0.30	0.87

The rest of parameters and then the thermal conductivity will be calculated using the Johansen approach as abovementioned.

3.2.6. Custom

Alternatively, the thermal conductivity can be determined with a set of inputted data points. Users can define a table of thermal conductivity values with either of the two dependences: a) temperature dependent, or b) water content dependent. The thermal conductivity changes depending on either temperature or water content.

3.3. Thermal Heat Capacity

The thermal volumetric heat capacity needs to be defined when the heat transfer is under transient state (i.e. the heat transfer rate is time dependent). Three types of thermal volumetric heat capacity are available in RS2: constant, Jame Newman, and custom.

The latent heat option is also available in RS2. The energy absorbed or released during phase change will be accounted for when latent heat is included. Normally, when using the effective latent heat method to account for phase changes (Eq 1.16), very small time steps should be utilized to accurately account for the frost front and phase change zones. RS2 uses an effective enthalpy compensation to account for the inaccuracy due to method of using effective latent method. By implementing the enthalpy compensation, the phase change front was accounted for accurately independently of time step size. More details can be found in verification documents.

It is noted that two types of heat capacity are involved depending on the material type: specific heat capacity which measures the heat capacity per mass, and volumetric heat capacity which measures per volume.

3.3.1. Constant

For the constant method, three values should be inputted to get a thermal volumetric heat capacity for soils: unfrozen volumetric heat capacity, frozen volumetric heat capacity, and frozen temperature. The thermal volumetric heat capacity will stay unchanged at its state.

Other than soils, the typical specific heat capacities for common materials are listed in Table 3.5 below.

Table 3.5 **Specific Heat Capacity Values of Various Materials (from U.S. Army Corps of Engineers)**

Material	Temperature (°F)	Specific heat capacity (Btu/lb·°F)
Aluminum	-27.4	0.20
Asbestos fibers	--	0.25
Concrete (avg. stone)	--	0.20
Concrete (dams)	--	0.22
Copper	44.6	0.20
Corkboard	91	0.43
	-19	0.29
Cork, granulated	--	0.42
Fiberglas board	111	0.24
	-22	0.19
Foamglas	-20	0.16
Glass block, expanded	112	0.18
Glass sheets	--	0.20
Glass wool	--	0.16
Ice	32	0.48
Iron (alpha)	44.6	0.11
Masonry	--	0.22
Mineral wool	--	0.22

Perlite, expanded	--	0.22
Polystyrene, cellular foam	--	0.27
Polyurethane foam	--	0.25
Sawdust	--	0.60
Snow	---	0.50
Steel	--	0.12
Straw	--	0.35
Water	--	1.00
Woods (avg.)	68	0.33
Woods fiberboard	148	0.34

3.3.2. Jame Newman

The Jame (1977) and Newman (1995) method estimates the volumetric heat capacity for soil with the value of soil specific heat capacity (c_s). The formula is given as,

$$c = \gamma_d c_s + c_w w_u \gamma_w + c_i w_i \gamma_i \quad (3.23)$$

Where c is the volumetric heat capacity, γ_d is the dry unit weight of soil, γ_w is the unit weight of water, γ_i is the unit weight of ice, c_s is the specific heat capacity of soil solids, c_w is the specific heat capacity of unfrozen water, c_i is the specific heat capacity of ice, w_u is the gravimetric unfrozen water content of the soil, and w_i is the gravimetric ice content of the soil.

3.3.3. Custom

Alternatively, the volumetric heat capacity can be determined with a set of inputted data points. Users can define a table of volumetric heat capacity with either of the two dependences: a) temperature, or b) water content. The volumetric heat capacity changes with either temperature or water content.

3.4. Thermal Soil Unfrozen Water Content

The unfrozen water content in soils is crucial during phase change, it needs to be calculated for a thermal analysis because of its major effect on thermal conductivity and heat capacity.

For a given soil, the unfrozen water content primarily depends on the negative temperature (e.g. the temperature below freezing point). The soil freezing characteristic curve (SFCC) is often derived to display the relationship between the unfrozen water content in soil and negative temperature, where the unfrozen water content in soil is graphed as a function of temperature.

3.4.1. Konrad

Mackenzie et al., (2006), Lebeau and Konrad (2009) estimated volumetric unfrozen water content of soils from SFCC with an exponential relationship,

$$\theta_u = (\theta_w - \theta_r) \exp \left[- \left(\frac{T - T_{ef}}{\zeta} \right)^2 \right] + \theta_r \quad \text{if } T < T_{ef} \quad (3.24)$$

$$\begin{aligned} m_2^i &= -2(\theta_w - \theta_r) \frac{T - T_{ef}}{\zeta^2} \exp \left[- \left(\frac{T - T_{ef}}{\zeta} \right)^2 \right] \\ &= -2(\theta_u - \theta_r) \frac{T - T_{ef}}{\zeta^2} \quad \text{if } T < T_{ef} \end{aligned} \quad (3.25)$$

$$\zeta = \frac{T_{ef} - T_{ep}}{2} \quad (3.26)$$

Where θ_u is the unfrozen water content, θ_w is the volumetric water content of unfrozen soil (at $T \geq T_{ep}$), θ_r is the residual water content below the solidus temperature ($T < T_{ep}$), T_{ef} is the soil freezing point, or liquidus temperature, in °C, T_{ep} is the temperature at the end of ice phase change, or solidus temperature, in °C, and ζ is an empirical parameter, approximated using eqn (3.26).

3.4.2. Tice and Anderson

Tice and Anderson method (1976) suggested that when temperature is below soil freezing point, the gravimetric unfrozen water content of the soil can be represented by a power curve, involving two characteristic parameters of each soil: α and β . The formulas are given as,

$$w_u = \begin{cases} w_w & \text{if } T > T_{ef} \\ \alpha |T|^\beta & \text{else} \end{cases} \quad (3.27)$$

Where w_u is the gravimetric unfrozen water content, w_w is the gravimetric water content of unfrozen soil, T is the temperature below freezing in degrees C, T_{ef} is the soil freezing point, and α and β are experimental parameters.

Based on eqn (3.27), the volumetric unfrozen water content (θ_u) and the slope of SFCC (m_2^i) can be obtained in eqn (3.28) and (3.29) respectively,

$$\theta_u = \begin{cases} \theta_w & \text{if } T > T_{ef} \\ \frac{\gamma_d}{\rho_w} \alpha |T|^\beta & \text{else} \end{cases} \quad (3.28)$$

$$m_2^i = \begin{cases} 0 & \text{if } T > T_{ef} \\ -\frac{\gamma_d}{\rho_w} \alpha \beta |T|^{(\beta-1)} & \text{else} \end{cases} \quad (3.29)$$

Where θ_u is the volumetric unfrozen water content, θ_w is the volumetric water content of unfrozen soil, γ_d is the soil dry density, ρ_w is the water density, and m_2^i is the slope of SFCC.

The characteristic parameters α and β accounts for the phase composition of frozen soils based on the specific surface area of the soil mineral grains. The typical α and β values for common soil types are provided in Table 3.6 below.

Table 3.6 Typical α and β values for common soils (Anderson and Tice, 1972)

Soil	Surface Area (m ² /g)	α	β
Basalt	6	0.0345	1.13
Rust	10	0.1105	0.8
West Lebanon gravel, <100 μ	18	0.0382	0.64
Limonite	26	0.0882	0.83
Fairbanks silt	40	0.0481	0.33
Dow Field silty clay	50	0.1035	0.61
Kaolinite	84	0.238	0.36
Suffield silty clay	140	0.1392	0.31
Hawaiian clay	382	0.3242	0.24
Wyoming bentonite	800	0.5599	0.29
Umiat bentonite	800	0.6755	0.34

3.4.3. Soil Water Content

Another method to acquire the unfrozen water content in soil is through the soil water content in hydraulics properties.

The suction (ψ) is calculated with,

$$\psi = \frac{L_w}{273.15} T_f \quad (3.30)$$

where ψ is the suction, L_w is the latent water, and T_f is the soil freezing point (in °C).

The suction is inputted into a hydraulic model (available models in RS2 include: simple, Fredlund and Xing, Van Genuchten, Brooks and Corey, and Gardner) to calculate the soil water content. The soil water content is then used to obtain the soil unfrozen water content.

See the [RS2 Seepage Analysis](#) topic for more information about hydraulic (permeability) models.

3.4.4. Custom

Alternatively, the thermal unfrozen water content in soils can be determined with a set of inputted data points. Users can define a table of unfrozen water content values as function of temperature.

3.4.5. Simple

Assume a previous stage with temperature T_0 (where $T_0 < T_f$), and a current stage with temperature T_1 , (where $T_1 > T_f$). $\theta_u = 0$ at T_0 , $\theta_u = \theta_w$ at T_1 . θ_u is the volumetric unfrozen water content, θ_w is the volumetric water content of unfrozen soil, T_f is the soil freezing point.

The region $T_0 < T_f < T_1$ is the zone of phase change. For the simple method, it is assumed that the phase change happens immediately between T_0 and T_1 . When temperature is below T_0 , $\theta_u = 0$; when temperature is above T_1 , $\theta_u = \theta_w$. A SFCC is not used in this method.

3.5. Thermal Constants

Other than required input variables, a number of parameters used in thermal calculations are constant values. These parameters can either be temperature independent (constant) or temperature dependent.

By default, the constant parameters are identified as temperature independent (constant). Their values are listed in Table 3.7 below.

Table 3.7 **Constant values (temperature independent) in thermal analysis**

Parameter Name (Unit)	Symbol	Value
Ice density (kN/m ³)	ρ_i	0.918
Thermal conductivity of water (kW/m/°C)	k_w	6.00E-04
Thermal conductivity of ice (kW/m/°C)	k_i	2.22E-03

Thermal conductivity of air (kW/m/°C)	k_a	2.50E-05
Specific heat capacity of water (kJ/ton/°C)	c_w	4181
Specific heat capacity of ice (kJ/ton/°C)	c_i	2108
Specific heat capacity of air (kJ/ton/°C)	c_a	1930
Latent Water (kJ/ton)	L_w	3.34E+05
Linear thermal expansion coefficient of water (1/°C)	α_w	2.10E-04
Linear thermal expansion coefficient of ice (1/°C)	α_i	5.00E-05
Freezing temperature of water (°C)	T_f	0

When the thermal constant option is switched to temperature dependent, the parameters at different temperatures will be applied according to Table 3.8 below.

Table 3.8 Temperature dependent constant values in thermal analysis

Temperature (°C)	Unit Weight (kg/m ³)	Thermal Conductivity (W/m-K)	Specific Heat Capacity (J/kg-K)
100	958.0	0.679	4.219E+06
95	962.0	0.677	4.213E+06
90	965.0	0.675	4.208E+06
85	968.0	0.673	4.203E+06
80	972.0	0.67	4.198E+06
75	975.0	0.667	4.194E+06
70	978.0	0.663	4.191E+06
65	980.0	0.659	4.188E+06
60	983.0	0.654	4.185E+06
55	986.0	0.649	4.183E+06
50	988.0	0.644	4.182E+06
45	990.2	0.637	4.181E+06
40	992.3	0.631	4.179E+06
35	994.1	0.623	4.178E+06
30	995.7	0.615	4.178E+06
25	997.1	0.607	4.180E+06
20	998.3	0.598	4.182E+06
15	999.2	0.589	4.186E+06
10	999.8	0.58	4.192E+06
5	1000.0	0.571	4.202E+06
0.01	999.8	0.561	4.217E+06

0	916.2	2.22	2.050E+06
-5	917.5	2.25	2.027E+06
-10	918.9	2.3	2.000E+06
-15	919.4	2.34	1.972E+06
-20	919.4	2.39	1.943E+06
-25	919.6	2.45	1.913E+06
-30	920.0	2.5	1.882E+06
-35	920.4	2.57	1.851E+06
-40	920.8	2.63	1.818E+06
-50	921.6	2.76	1.751E+06
-60	922.4	2.90	1.681E+06
-70	923.3	3.05	1.609E+06
-80	924.1	3.19	1.536E+06
-90	924.9	3.34	1.463E+06
-100	925.7	3.48	1.389E+06

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