RS2 THEORY

Coupled Consolidation

July 2023 modified

Rocscience

Biot Consolidation consists of a theory of coupled solid-fluid interaction; the Biot consolidation equations have been implemented in RS2 in order to model coupled consolidation. The following is a summary of the coupled equations, as described in “The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media” by Lewis, R.W [1].

Biot Consolidation

In the Biot theory, the soil skeleton is treated as a porous elastic solid, with laminar pore fluid coupled with it. This coupling is accomplished through conditions of compressibility and continuity. Biot’s governing equation is:

$$\int_{\Omega} (\boldsymbol{L}\boldsymbol{N}_u)^T \boldsymbol{\sigma}\, d\Omega = \int_{\Omega} \boldsymbol{N}_u^T \rho \boldsymbol{g}\, d\Omega + \int_{\Gamma_N^q} \boldsymbol{N}_u^T \bar{\boldsymbol{t}}\, d\Gamma \quad (1)$$

$$\int_{\Omega} (\nabla \boldsymbol{N}_p)^T \frac{\boldsymbol{k}}{\mu^w} \nabla \boldsymbol{N}_p \bar{\boldsymbol{p}}^w\, d\Omega - \int_{\Omega} (\nabla \boldsymbol{N}_p)^T \frac{\boldsymbol{k}}{\mu^w} \rho^w \boldsymbol{g}\, d\Omega + \int_{\Omega} \boldsymbol{N}_p^T \alpha \boldsymbol{m}^T \boldsymbol{L}\boldsymbol{N}_u \frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \int_{\Omega} \boldsymbol{N}_p^T \left(\frac{\alpha - n}{K_s} + \frac{n}{K_w}\right) \boldsymbol{N}_p \frac{\partial \bar{\boldsymbol{P}}^w}{\partial t}\, d\Omega + \int_{\Gamma_w^q} \boldsymbol{N}_p^T \frac{q^w}{\rho^w}\, d\Gamma = 0 \quad (2)$$

Where

$\boldsymbol{N}_u$ is the shape function in terms of the displacement, $\boldsymbol{L}$ is the differential operator, ρ is the density, $\boldsymbol{g}$ is the gravitational acceleration, $\bar{\boldsymbol{t}}$ is the direction traction of the surface, Γ_N^q is the surface of interest, Ω is the domain of interest, Γ is the boundary of domain of interest, $\boldsymbol{N}_p$ is the shape function in terms of pore pressure, $\boldsymbol{k}$ is the intrinsic permeability, μ^w is the viscosity, $\bar{\boldsymbol{p}}^w$ is the vectors of the nodal values of the unknowns, ρ^w is the intrinsic density of water, n is the porosity, K_s is the bulk modulus of the grain material. In RS2, K_s was taken as 69 GPa. $K_w = 1/C_w$ is the bulk modulus of water , $\bar{\boldsymbol{P}}^w$ is the water pressure, q^w is the imposed mass flux normal to the boundary, Γ_w^q is the surface of interest,

$$\boldsymbol{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

$\boldsymbol{\sigma}$ is the stress tensor

$$\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\}^T$$

α is the Biot's constant calculated as:

$$\alpha = 1 - \frac{K_t}{K_s}$$

Where K_t is the bulk modulus of drain porous medium taken as:

$$K_t = \frac{E}{3(1-2\upsilon)}$$

υ is the Poisson's ratio

$\boldsymbol{m}$ is the vector

$$\boldsymbol{m}^T = [1, 1, 1, 0, 0, 0]^T$$

The gradient operator used is

$$\nabla = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}^T$$

These equations are rewritten as

$$\int_\Omega \boldsymbol{B}^T \boldsymbol{\sigma}'' d\Omega - \mathbf{Q}\overline{\boldsymbol{p}}^w = \boldsymbol{f}^u \qquad (3)$$

$$\boldsymbol{H}\overline{\boldsymbol{p}}^w + \boldsymbol{Q}^T \frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \boldsymbol{S}\frac{\partial \overline{\boldsymbol{p}}^w}{\partial t} = \boldsymbol{f}^p \qquad (4)$$

where

$\boldsymbol{B} = \boldsymbol{L}\boldsymbol{N}_u$ is the strain operator

$\boldsymbol{Q} = \int_\Omega \boldsymbol{B}^T \alpha \boldsymbol{m} \boldsymbol{N}_p d\Omega$ is the coupling matrix

$\boldsymbol{H} = \int_{\Omega} \left(\nabla \boldsymbol{N}_p\right)^T \frac{\boldsymbol{k}}{\mu^w} \nabla \boldsymbol{N}_p d\Omega$ is the permeability matrix

$\boldsymbol{S} = \int_{\Omega} \boldsymbol{N}_p^T \left(\frac{\alpha - n}{\boldsymbol{K}_s} + \frac{n}{\boldsymbol{K}_w}\right) \boldsymbol{N}_p d\Omega$ is the compressibility matrix

The right-hand terms in equations (3) and (4) are given by

$$\boldsymbol{f}^u = \int_{\Omega} \boldsymbol{N}_u^T [\rho^s(n-1) + \rho^w n] \boldsymbol{g} d\Omega + \int_{\Gamma_w^q} \boldsymbol{N}_u^T \bar{\boldsymbol{t}}\, d\Gamma$$

$$\boldsymbol{f}^p = \int_{\Omega} \left(\nabla \boldsymbol{N}_p\right)^T \frac{\boldsymbol{k}}{\mu^w} \rho^w \boldsymbol{g} d\Omega - \int_{\Gamma_w^q} \boldsymbol{N}_p^T \frac{q^w}{\rho^w}\, d\Gamma \qquad (5)$$

The first term in equation (3) represents the internal force

$$\boldsymbol{P}(\overline{\boldsymbol{u}}) = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{\sigma}''\, d\Omega \qquad (6)$$

For the isotropic linear elastic case, the constitutive relationship can be written as

$$\boldsymbol{\sigma}'' = \boldsymbol{D}_e \boldsymbol{\varepsilon} = \boldsymbol{D}_e \boldsymbol{L} \boldsymbol{N}_u \overline{\boldsymbol{u}} = \boldsymbol{D}_e \boldsymbol{B} \overline{\boldsymbol{u}} \qquad (7)$$

The internal force can therefore be written as

$$\boldsymbol{P}(\overline{\boldsymbol{u}}) = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{\sigma}''\, d\Omega = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{D}_e \boldsymbol{B}\, d\Omega\, \overline{\boldsymbol{u}} = \boldsymbol{K}_e \overline{\boldsymbol{u}} \qquad (8)$$

Where $\boldsymbol{K}_e = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{D}_e \boldsymbol{B}\, d\Omega$ is the linear elastic stiffness matrix, which is symmetric in form [3]. However, in problems where the solid-phase behaviour is non-linear, only the tangential stiffness matrix $\boldsymbol{K}_T$ can be defined.

$$\boldsymbol{K}_T = \frac{\partial \boldsymbol{P}(\overline{\boldsymbol{u}})}{\partial \overline{\boldsymbol{u}}} = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{D}_T \boldsymbol{B}\, d\Omega$$

$$\frac{\partial \boldsymbol{P}(\overline{\boldsymbol{u}})}{\partial t} = \frac{\partial \boldsymbol{P}(\overline{\boldsymbol{u}})}{\partial \overline{\boldsymbol{u}}} \frac{\partial \overline{\boldsymbol{u}}}{\partial t} = \boldsymbol{K}_T \frac{\partial \overline{\boldsymbol{u}}}{\partial t} \qquad (9)$$

Integration of the above matrices usually requires numerical techniques. A standard method is that of Gaussian quadrature [3], where the integrands are evaluated at specific points of the element then weighted and summed. The procedure is carried out in terms of a set of local coordinates [3].

Since the discretization in space has been carried out, equation (3) and (4) now represent a set of ordinary differential equations in time. For convenience, the equations are written in the following form, with the assumption of linear elastic behaviour of the solid skeleton:

$$\begin{bmatrix} \boldsymbol{K}_e & -\boldsymbol{Q} \\ \boldsymbol{0} & \boldsymbol{H} \end{bmatrix} \begin{Bmatrix} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{p}}^w \end{Bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{Q}^T & \boldsymbol{S} \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{p}}^w \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f}^u \\ \boldsymbol{f}^p \end{Bmatrix} \qquad (10)$$

By carrying time differentiating equation (3) and by multiplying one set of equations by -1. This procedure shifts the component matrices horizontally in the global matrix and is common in the analysis of consolidation problems. It will yield the following set of equations:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H} \end{bmatrix} \begin{Bmatrix} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{p}}^w \end{Bmatrix} + \begin{bmatrix} -\boldsymbol{K}_T & \boldsymbol{Q} \\ \boldsymbol{Q}^T & \boldsymbol{S} \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{p}}^w \end{Bmatrix} = \begin{Bmatrix} -\frac{d}{dt}\boldsymbol{f}^u \\ \boldsymbol{f}^p \end{Bmatrix} \tag{11}$$

Time differentiation and successive integration is a possible way of introducing computationally non-linear behaviour. This is the reason why K_T appears in equation (7).

Evaluating the matrices in equation (6) at time $t_{n+\theta}$ yields

$$\begin{bmatrix} \theta\boldsymbol{K}_e & -\theta\boldsymbol{Q} \\ \boldsymbol{Q}^T & \boldsymbol{S} + \Delta t\theta\boldsymbol{H} \end{bmatrix}_{n+\theta} \begin{Bmatrix} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{p}}^w \end{Bmatrix}_{n+1} = \begin{bmatrix} (\theta-1)\boldsymbol{K}_e & (1-\theta)\boldsymbol{Q} \\ \boldsymbol{Q}^T & \boldsymbol{S} - (1-\theta)\Delta t\boldsymbol{H} \end{bmatrix}_{n+\theta} \begin{Bmatrix} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{p}}^w \end{Bmatrix}_n + \begin{Bmatrix} \boldsymbol{f}^n \\ \Delta t\boldsymbol{f}^p \end{Bmatrix}_{n+\theta} \tag{12}$$

The complete set of equations (8), may be used to determine the values of $\bar{u}$ and $\bar{p}^w$ at any time relative to their initial values. It can easily be verified.

Discretization of the governing equations for the consolidation of partially saturated soils

When standard finite element discretization techniques are applied to the equilibrium equation along with the boundary condition, they yield

$$\int_\Omega (\boldsymbol{L}\boldsymbol{N}_u)^T\boldsymbol{\sigma}\, d\Omega = \int_\Omega \boldsymbol{N}_u^T\rho\boldsymbol{g}\, d\Omega + \int_{\Gamma_u^q} \boldsymbol{N}_u^T\bar{\boldsymbol{t}}\, d\Gamma \tag{13}$$

Introduction of the effective stress principal results in

$$\int_\Omega \boldsymbol{B}^T\boldsymbol{\sigma}^n\, d\Omega - \boldsymbol{Q}\overline{\boldsymbol{p}}^w = \boldsymbol{f}^u \tag{14}$$

Where

$$\boldsymbol{Q} = \int_\Omega \boldsymbol{B}^T\chi\alpha\boldsymbol{m}\boldsymbol{N}_p\, d\Omega \tag{15}$$

is the coupling matrix, where χ is calculated based on the unsaturated effective stress method as mentioned in the “RS2: Soil Behaviors in Unsaturated Zones” document.

and the load vector is

$$\boldsymbol{f}^n = \int_\Omega \boldsymbol{N}_u^T[\rho^s(n-1) + S_w n\rho^w]\boldsymbol{g}\, d\Omega + \int_{\Gamma_u^q} \boldsymbol{N}_u^T\bar{\boldsymbol{t}}\, d\Gamma \tag{16}$$

Before discretizing the equation, we take into account that

$$n\frac{\partial S_w}{\partial t} = n\frac{\partial S_w}{\partial p^w}\frac{\partial p^w}{\partial t} = C_s\frac{\partial p^w}{\partial t} \tag{17}$$

where $C_s = n\partial S_w/\partial p^w$ is the specific moisture content. Hence the mass balance equation becomes

$$\left[\frac{\alpha - n}{K_s}S_w\left(S_w + \frac{C_s}{n}p^w\right) + \frac{nS_w}{K_w} + C_s\right]\frac{\partial p^w}{\partial t} + \alpha S_w\boldsymbol{m}^T\boldsymbol{L}\frac{\partial\boldsymbol{u}}{\partial t} + \nabla^T\left[\frac{\boldsymbol{k}k^{rw}}{\mu^w}(-\nabla p^w + \rho^w\boldsymbol{g})^w\right] = 0 \tag{18}$$

The discretization in space is carried out exactly as for the fully saturated case and yields

$$\boldsymbol{H}\overline{\boldsymbol{p}}^{w} + \overline{\boldsymbol{Q}}^{T}\frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \boldsymbol{S}\frac{\partial \overline{\boldsymbol{p}}^{w}}{\partial t} = \boldsymbol{f}^{p} \tag{19}$$

where the permeability matrix is

$$\boldsymbol{H} = \int_{\Omega} \left(\boldsymbol{\nabla N}_{p}\right)^{T} \frac{\boldsymbol{k}k^{rw}}{\mu^{w}} \nabla \boldsymbol{N}_{p} \, d\Omega \tag{20}$$

the compressibility matrix is

$$\boldsymbol{S} = \int_{\Omega} \boldsymbol{N}_{p}^{T} \left[\frac{\alpha - n}{K_{s}} S_{w} \left(S_{w} + \frac{C_{s}}{n} p^{w}\right) + \frac{nS_{w}}{K_{w}} + C_{s}\right] N_{p} \, d\Omega \tag{21}$$

the coupling matrix is

$$\overline{\boldsymbol{Q}} = \int_{\Omega} \boldsymbol{B}^{T} S_{w} \alpha \boldsymbol{m} \boldsymbol{N}_{p} \, d\Omega \tag{22}$$

and the right-hand term (flow vector) is

$$\boldsymbol{f}^{p} = \int_{\Omega} \left(\nabla \boldsymbol{N}_{p}\right)^{T} \frac{\boldsymbol{k}k^{rw}}{\mu^{w}} \rho^{w} \boldsymbol{g} \, d\Omega - \int_{\Gamma_{u}^{\sigma}} \boldsymbol{N}_{p}^{T} \frac{q^{w}}{\rho^{w}} d\Gamma \tag{23}$$

The structure of the two discretized equations, equation (11) and (16), is very similar to that of the fully saturated case. The resulting system of equations is formally identical to equation (9) except that the component matrices are now defined in this section. Now time differentiation of the equilibrium equation does not restore symmetry: the two coupling matrices are no longer the same. However, symmetry can be restored by using staggered solution procedures.

Furthermore, the resulting system of equations is non-linear, even for the case of linear elastic solid behaviour. In general, this requires iterations within each time step. The numerical properties of the time discretization require further examination.

Note that if Bishop [2] method is used to calculate χ, then equation (12) is equal to equation (19), so $\boldsymbol{Q} = \overline{\boldsymbol{Q}}$.

References

[1] Lewis, R .W. (1998). The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media, West Sussex, England: John Wiley and Sons Ltd.

[2] Bishop, A. W. (1959). The principle of effective stress.Tecknish Ukeblad 106, 859-863.

[3] Zienkiewiez, O. C., Chan, A. H. C., Pastor, M., Paul, D. K. and Shiomi, T. (1990). Statis and dynamic behaviour of soils: a rational approach to quantitative solutions, I. Fully saturated problems. Proc. R. Soc. Lond. A, 429, 285-309.