

# Strains in Phase<sup>2</sup>

## Quantifying Strains

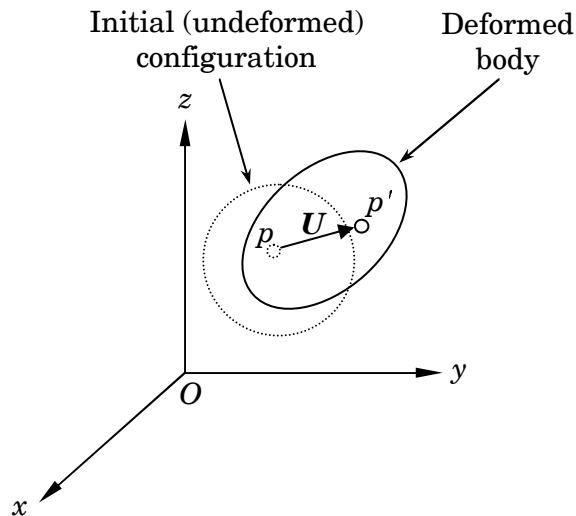


Fig. 1a: Deformation of a continuum body. The displacement vector  $\mathbf{U}$  links the position of point  $p$  in the undeformed body to its position  $p'$  in the deformed configuration.

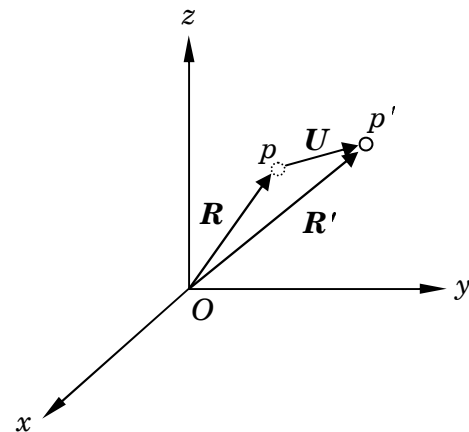


Fig. 1b: The displacement vector  $\mathbf{U}$  has components  $(u, v, w)$  and is equal to  $\mathbf{R}' - \mathbf{R}$ .

Figure 1 above illustrates how the strains that develop in a deformed continuum body can be described mathematically. The deformations can be characterized by assigning a displacement vector to each point of the body. The set of displacement vectors for all points is known as a displacement vector field. Knowledge of this field completely defines the deformed state of the body.

The displacement field may be composed of two parts: rigid body motion and straining. In order to separate straining from rigid motion, we first define a displacement gradient matrix,  $\nabla \mathbf{U}$ , which comprises partial derivatives of  $\mathbf{U}$ .

$$\nabla \mathbf{U} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

These partial derivatives are not affected by rigid translations. They are however influenced by rigid rotations. Elimination of the influence of rotations requires use of what is known as the symmetric part of  $\nabla U$ . The symmetric part,  $\boldsymbol{\varepsilon}$  (also known as the strain matrix), has the definition

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla U + (\nabla U)^T] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \boldsymbol{\varepsilon}_{xy} & \boldsymbol{\varepsilon}_{xz} \\ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} & \boldsymbol{\varepsilon}_{yz} \\ \boldsymbol{\varepsilon}_{zx} & \boldsymbol{\varepsilon}_{zy} & \boldsymbol{\varepsilon}_{zz} \end{bmatrix}.$$

The symmetry of the strain matrix can be seen from the fact that

$$\begin{aligned} \boldsymbol{\varepsilon}_{xy} &= \boldsymbol{\varepsilon}_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\ \boldsymbol{\varepsilon}_{xz} &= \boldsymbol{\varepsilon}_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \text{ and} \\ \boldsymbol{\varepsilon}_{yz} &= \boldsymbol{\varepsilon}_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right). \end{aligned}$$

The diagonal components of  $\boldsymbol{\varepsilon}$ , which are known as *extensional strains*, describe changes in length per unit length along the coordinate directions. The geometric interpretation of the off-diagonal components, which are known as *shear strains* or at times *tensorial shear strains*, will be discussed later.

$\boldsymbol{\varepsilon}$  is sometimes referred to as the *small strain* matrix. What this means is that the components of  $\boldsymbol{\varepsilon}$  correctly capture straining so long as the components of  $\nabla U$  are much smaller in magnitude than 1.

## Shear Strain versus Engineering Shear Strain

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There are two definitions of shear strain. The first one, which we have already encountered is known as *tensorial shear strain* or simply *shear strain*. This definition is more favoured by those working in the theories of elasticity and plasticity.

The second definition is *engineering shear strain*. It is more prevalent in engineering applications. Mathematically, engineering shear strains are the expressions in the parentheses in the off-diagonal terms of the strain tensor given above.

Let us examine the definition of the shear strain  $\varepsilon_{xy}$ , as an example –

$$\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right).$$

The corresponding engineering shear strain,  $\gamma_{xy}$ , is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\varepsilon_{xy}.$$

### Geometric Interpretation of Engineering Shear Strain and Shear Strain

Engineering shear strain and shear strain have geometric interpretations. We shall look at this interpretation through the example of the shear deformations of an infinitesimal square element at point  $p$  in the  $x$ - $y$  plane (Figure 2).

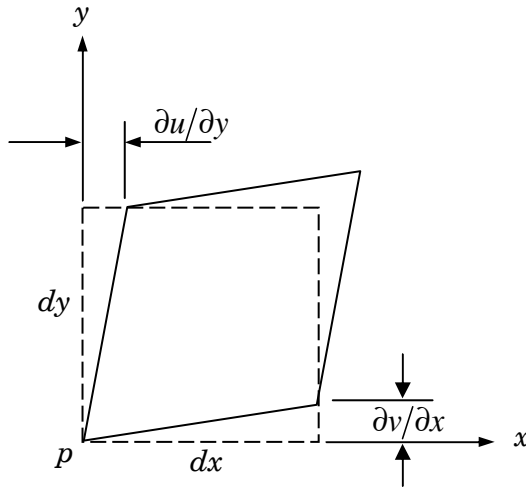


Fig. 2a: The  $\varepsilon_{xy}$  shear strain component of the strain tensor is the average of two strains  $\partial v/\partial x$  and  $\partial u/\partial y$ .

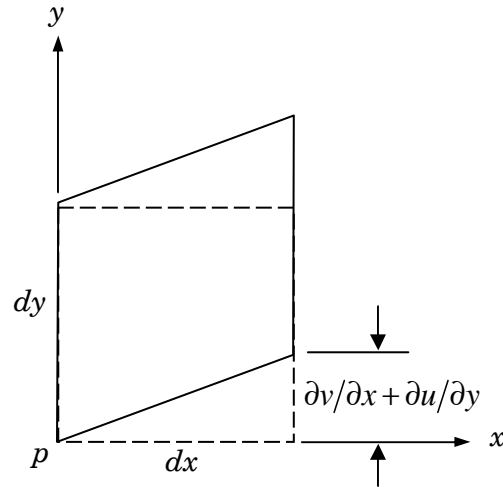


Fig. 2b: The  $\gamma_{xy}$  engineering shear strain is the sum of two strains  $\partial v/\partial x$  and  $\partial u/\partial y$ .

From Figure 2, it can be seen that shear deformations change the angle between the sides of the element, and shear strain quantifies this change. The shear strain

$\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$  is the average of the shear strain strains  $\partial v/\partial x$  and  $\partial u/\partial y$ , while

the engineering shear strain  $\gamma_{xy} = \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$  is the **total** shear strain in the  $x$ - $y$  plane.

## ***Phase*<sup>2</sup> Conventions**

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The shear strains reported in *Phase*<sup>2</sup> Interpret are actually engineering shear strains. For example, the strain denoted “exy” in Interpret is the engineering shear strain  $\gamma_{xy}$ .

The convention for extensional strains in *Phase*<sup>2</sup> is as follow:

- Extensional strain is positive if an element has shortened along the coordinate direction of interest
- Extensional strain is negative if an element has lengthened along the coordinate direction of interest

## **Principal Strains**

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Because the strain matrix  $\boldsymbol{\varepsilon}$  is symmetric, a set of orthogonal directions can be determined for which only lengthening or shortening occurs; in this coordinate system no shear takes place.

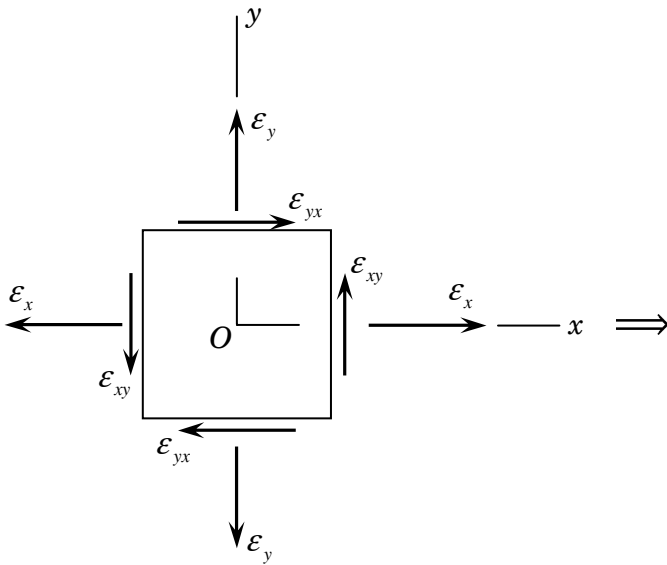


Fig. 3a: Strain components in original coordinate system

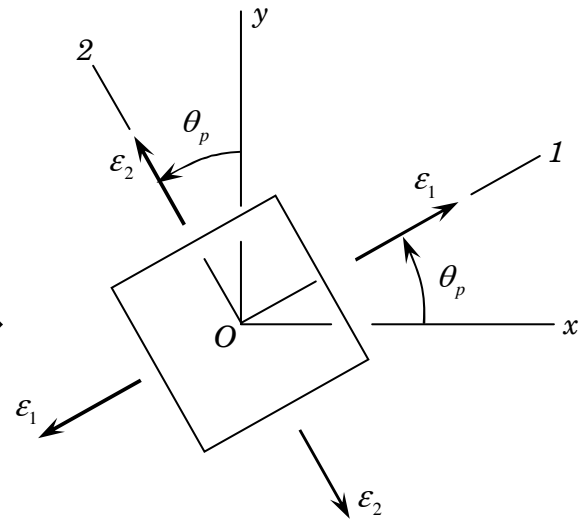


Fig. 3b: Strain components transformed to principal directions

In the case of two-dimensional analysis (Figs. 3a and 3b), the principal strains  $\varepsilon_1$  and  $\varepsilon_2$  can be determined from the expressions:

$$\varepsilon_1 = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad \text{and}$$

$$\varepsilon_2 = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}.$$

The angle of rotation which yields the principal strain directions is

$$\theta_p = \frac{1}{2} \tan^{-1} \left( \frac{2\varepsilon_{xy}}{\varepsilon_x - \varepsilon_y} \right).$$

## Volumetric Strain

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Although the extensional and shear strain components of  $\varepsilon$  completely define the deformed state of a body, it is also possible to measure other characteristic values of strain. Volumetric strain is a measure of the change in volume per unit volume of the material. It is defined as the sum of the extensional strains. In plane strain analysis, volumetric strain is

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} = \varepsilon_1 + \varepsilon_2.$$

## Maximum Shear Strain

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Just as there is an orientation of axes in which shear strains disappear, there is also a coordinate transformation that gives the direction in which the maximum shear strains occurs.

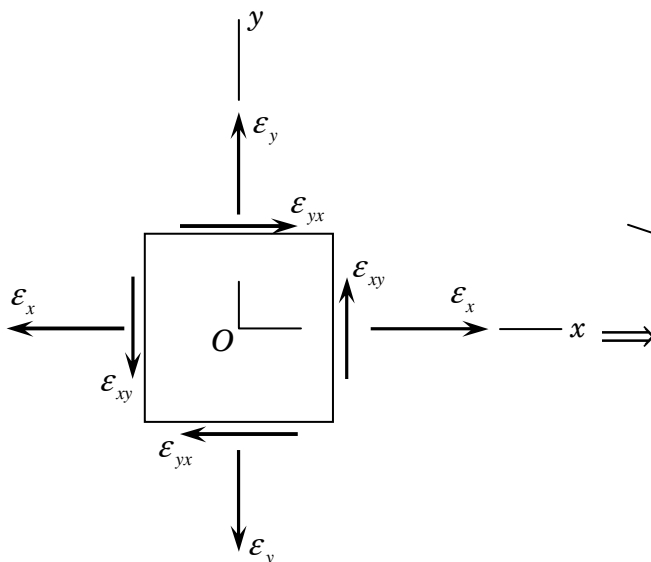


Fig. 4a: Strain components in original coordinate system

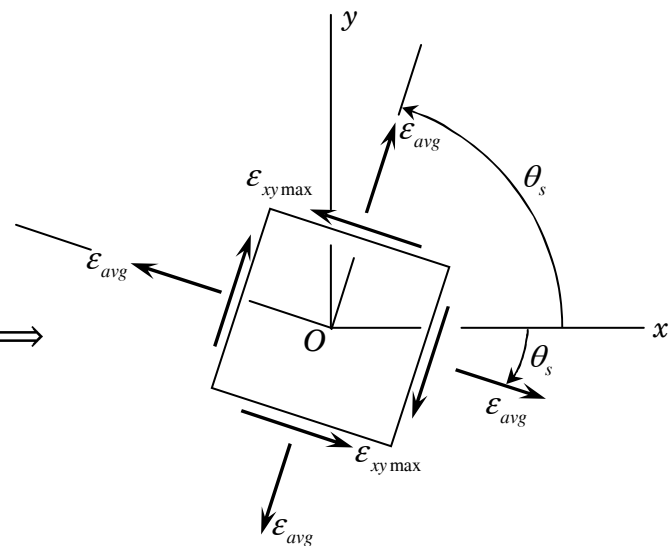


Fig. 4b: Maximum shear strain. Note that the extensional strains in the transformed system equal an average strain,  $\varepsilon_{avg}$ .

In plane strain analysis, the maximum shear strain is simply

$$\varepsilon_{xy \max} = \frac{\varepsilon_1 - \varepsilon_2}{2} .$$

The rotation angle at which this strain can be calculated as

$$\theta_s = \frac{1}{2} \tan^{-1} \left( -\frac{\varepsilon_x - \varepsilon_y}{2\varepsilon_{xy}} \right) = \theta_p \pm 45^\circ .$$

### **Calculating Principal Strains from Volumetric and Maximum Shear Strains**

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For two-dimensional analysis, given volumetric strain and maximum shear strain, the pair of principal strains can be calculated using the following equations:

$$\varepsilon_1 = \frac{\varepsilon_v}{2} + \varepsilon_{xy \max} , \text{ and}$$

$$\varepsilon_2 = \frac{\varepsilon_v}{2} - \varepsilon_{xy \max} .$$

### **References**

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1. Davis, R.O. and Selvadurai, A.P.S. 2002. *Plasticity and Geomechanics*, Cambridge University Press, Cambridge.
2. Engineering Fundamentals – efunda. “Mechanics of Materials: Strain.” [www.efunda.com/formulae/solid\\_mechanics/mat\\_mechanics/strain.cfm](http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/strain.cfm).
3. Landis, C.M. 2007. “Tensor operations. Max shear stress, 2D vs 3D,” Class Notes for Advanced Mechanics of Materials Course, Rice University.