RS3

Plate Elements

Theory Manual

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Introduction

Plate element formulations can be based on either the classical Kirchhoff thin plate theory or the Mindlin-Reissner plate theories. The Kirchhoff theory, which extends the Euler-Bernoulli beam theory to plates, neglects transverse shear deformations and requires at least C^1 continuity, meaning that higher-order elements are necessary. In contrast, the Mindlin theory accounts for transverse shear effects, similar to the Timoshenko theory for beams, and can use C^0 continuity for lateral displacements and two independent rotations (Cook et al., 1989). Because RS3 employs 3-noded and 6-noded plate elements, the Mindlin elements are used to take advantage of these properties. In the following section, RS3’s Mindlin element formulation is presented.

1. Linear Momentum and Weak Form Equations

The balance of linear momentum (equilibrium equations) for plates begins with the following generalized three-dimensional field equations subject to relevant Dirichlet and Neumann boundary conditions

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g} - \rho \ddot{\boldsymbol{u}} = \boldsymbol{0} \quad \text{in} \quad \Omega = (\partial\Omega_t \cup \partial\Omega_u) \in \mathbb{R}^3 \tag{1}$$

where Ω is the spatial domain, $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j$, ρ is the density, $\boldsymbol{g}$ denotes the vector of gravitational acceleration, and $\ddot{\boldsymbol{u}}$ is the acceleration ($\boldsymbol{u}$ is the displacement). Neglecting tractions on the Neumann boundary ($\partial\Omega_t$) and acceleration in the spatial domain "*for convenience and simplicity in presenting the equations"*, the Voigt notation of the weak form of equation (1), using the Galerkin method and considering infinitesimal deformations, leads to

$$\psi(\{\boldsymbol{u}\},\{\delta\boldsymbol{u}\}) = \int_\Omega (\{\delta\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}\} - \rho\{\delta\boldsymbol{u}\}^T\{\boldsymbol{g}\})d\Omega = 0 \tag{2}$$

where $\{\delta\boldsymbol{u}\}^T = \{\delta u(x,y,z), \delta v(x,y,z), \delta w(x,y,z)\}$ is the variation of displacements in x, y, and z directions, and

$$\{\delta\boldsymbol{\varepsilon}\}^T = \left\{\delta\varepsilon_{xx}, \delta\varepsilon_{yy}, \delta\varepsilon_{zz}, \delta\gamma_{xy}, \delta\gamma_{xz}, \delta\gamma_{yz}\right\}; \quad \{\boldsymbol{\sigma}\}^T = \left\{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}\right\}$$

2. Mindlin Plate: Kinematics and Kinetics

Assuming (x,y,z) as the local Cartesian coordinate system, where z is perpendicular to the flat Mindlin plate with thickness t, the domain of interest, Ω, is specified as

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 |\, z \in [-t/2\,, t/2]\ \&\ (x,y) \in \mathbb{R}^2\} \tag{3}$$

The kinematics of Mindlin plates require the lateral deformation, w, to be independent of the z-direction, and the normal vector to the surface $z = 0$ remains straight but not necessarily perpendicular after deformation occurs (Owen and Hinton, 1980). Accordingly, the vector of displacement $\{\boldsymbol{u}\}$ is

$$\{\boldsymbol{u}\}^T = \{u(x,y,z), v(x,y,z), w(x,y,z)\} = \left\{-z\theta_x(x,y), -z\theta_y(x,y), w(x,y)\right\} \tag{4}$$

where θ_x and θ_y are the rotations of the normal vector to the surface $z = 0$ in the xz and yz planes, respectively.

In Mindlin plates, the normal stress σ_{zz} is assumed to be zero, and the vector $\{\boldsymbol{\sigma}\}$ is described by five components. Additionally, the vector $\{\boldsymbol{\sigma}\}$ can be decomposed into flexural-shear and membrane stresses as given

$$\{\boldsymbol{\sigma}\}^T = \left\{\left\{\boldsymbol{\sigma}_{flexural}\right\}^T, \{\boldsymbol{\sigma}_{shear}\}^T\right\} + \{\boldsymbol{\sigma}_{membrane}\}^T \tag{5}$$

where

$$\{\boldsymbol{\sigma}_{membrane}\}^T = \{\boldsymbol{\sigma}_m\}^T = \left\{\sigma_{xx}^m, \sigma_{yy}^m, 0,0,0\right\}$$

$$\left\{\boldsymbol{\sigma}_{flexural}\right\}^T = \left\{\boldsymbol{\sigma}_f\right\}^T = \left\{\sigma_{xx}, \sigma_{yy}, \tau_{xy}\right\}$$

$$\{\boldsymbol{\sigma}_{shear}\}^T = \{\boldsymbol{\sigma}_s\}^T = \left\{\tau_{xz}, \tau_{yz}\right\}$$

In equation (5), the superscript "m" denotes the membrane stress components, while the stress components without a superscript correspond to the flexural-shear counterparts. Since w is independent to z (equation (4)), $\delta\varepsilon_{zz}$ is equal to zero. Therefore, the vector $\{\delta\boldsymbol{\varepsilon}\}$ can be restated as

$$\{\delta\boldsymbol{\varepsilon}\}^T = \left\{\left\{\delta\boldsymbol{\varepsilon}_f\right\}^T, \{\delta\boldsymbol{\varepsilon}_s\}^T\right\} \tag{6}$$

where

$$\left\{\delta\boldsymbol{\varepsilon}_f\right\}^T = \left\{\delta\varepsilon_{xx}, \delta\varepsilon_{yy}, \delta\gamma_{xy}\right\}$$

$$\{\delta\boldsymbol{\varepsilon}_s\}^T = \left\{\delta\gamma_{xz}, \delta\gamma_{yz}\right\}$$

Using the kinematics expressed in equation (4), $\left\{\delta\boldsymbol{\varepsilon}_f\right\}$ and $\{\delta\boldsymbol{\varepsilon}_s\}$ are

$$\begin{gathered}\left\{\delta\boldsymbol{\varepsilon}_f\right\}^T = z\left\{-\partial(\delta\theta_x)/\partial x\,, -\partial\left(\delta\theta_y\right)/\partial y\,, -\left(\partial(\delta\theta_x)/\partial y + \partial\left(\delta\theta_y\right)/\partial x\right)\right\} = z\left\{\delta\hat{\boldsymbol{\varepsilon}}_f\right\}^T \\ \{\delta\boldsymbol{\varepsilon}_s\}^T = \left\{(\partial(\delta w)/\partial x) - \delta\theta_x, (\partial(\delta w)/\partial y) - \delta\theta_y\right\}\end{gathered} \tag{7}$$

3. Mindlin Plate: Weak Form Equations

To obtain the weak-form equation for Mindlin plates, equations (5,6) can be inserted into equation (2), resulting in

$$\psi(\{\boldsymbol{u}\},\{\delta\boldsymbol{u}\}) = \int_x\int_y\int_{-t/2}^{t/2}\left(z\left\{\delta\hat{\boldsymbol{\varepsilon}}_f\right\}^T\left\{\boldsymbol{\sigma}_f\right\} + \{\delta\boldsymbol{\varepsilon}_s\}^T\{\boldsymbol{\sigma}_s\} + \{\delta\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}_m\} - \rho\{\delta\boldsymbol{u}\}^T\{\boldsymbol{g}\}\right)dzdydx = 0 \tag{8}$$

Using equation (4) and integrating along the z-direction, equation (8) leads to

$$\begin{gathered}\psi(\{\boldsymbol{u}\},\{\delta\boldsymbol{u}\}) = \int_x\int_y\left(\left\{\delta\hat{\boldsymbol{\varepsilon}}_f\right\}^T\{\boldsymbol{M}\} + \{\delta\boldsymbol{\varepsilon}_s\}^T\{\boldsymbol{Q}\} - \{\delta\boldsymbol{u}\}^T\{\boldsymbol{q}\}\right)dydx \\ + \int_x\int_y\int_{-t/2}^{t/2}\{\delta\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}_m\}dzdydx = 0\end{gathered} \tag{9}$$

where $\{\boldsymbol{M}\}$, $\{\boldsymbol{Q}\}$, and $\{\boldsymbol{q}\}$ are the bending moments, shear forces, and lateral distributed loads acting on the plate which are defined as

$$\begin{gathered}\{\boldsymbol{M}\}^T = \left\{M_{xx}, M_{yy}, M_{xy}\right\} = \int_{-t/2}^{t/2} z\left\{\boldsymbol{\sigma}_f\right\}^T dz \\ \{\boldsymbol{Q}\}^T = \left\{Q_x, Q_y\right\} = \int_{-t/2}^{t/2}\{\boldsymbol{\sigma}_s\}^T dz \\ \{\boldsymbol{q}\}^T = \int_{-t/2}^{t/2}\rho\{\boldsymbol{g}\}^T dz\end{gathered} \tag{10}$$

4. Mindlin Plate: Constitutive Equations

Rate-independent constitutive equations relating changes in stresses to changes in infinitesimal strains are used for Mindlin plates. For isotropic elastic materials, these equations are as follows.

The constitutive equations for isotropic elastic Mindlin plates require defining the relationships between the flexural, shear, and membrane stresses and their corresponding strain measures. This begins by applying Hooke's law for isotropic elastic materials. By omitting the zero component $\delta\sigma_{zz}$ from the rate-form constitutive equation, and considering that w is independent to the z-driection (i.e. $\delta\varepsilon_{zz} = 0$; refer to equation (4)), the general constitutive relation for stresses is

$$\{\delta\boldsymbol{\sigma}\} = [\mathbb{D}]\{\delta\boldsymbol{\varepsilon}\} \tag{11}$$

where

$$[\mathbb{D}] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & (1-\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & (1-\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{12}$$

where E and ν are Young's modulus and Poisson's ratio. Accordingly, the constitutive equation for the change in membrane stress is as follows

$$\{\delta\boldsymbol{\sigma}_m\} = [\mathbb{D}_m]\{\delta\boldsymbol{\varepsilon}\} \tag{13}$$

where

$$[\mathbb{D}_m] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

In order to obtain the constitutive relations for the flexural and shear stresses, the changes in $\{\boldsymbol{\sigma}_f\}$ and $\{\boldsymbol{\sigma}_s\}$ in equation (8) can be analyzed using the general constitutive equation (11) and the kinematic relation (6). Accordingly, the first two terms of equation (8) are

$$\begin{aligned} &\int_{\boldsymbol{x}}\int_y\int_{-t/2}^{t/2}\left(z\{\delta\hat{\boldsymbol{\varepsilon}}_f\}^T\{\delta\boldsymbol{\sigma}_f\} + \{\delta\boldsymbol{\varepsilon}_s\}^T\{\delta\boldsymbol{\sigma}_s\}\right)dzdydx \\ &= \int_{\boldsymbol{x}}\int_y\int_{-t/2}^{t/2}\left(z\{\delta\hat{\boldsymbol{\varepsilon}}_f\}^T[\mathbb{D}_f]\{\delta\boldsymbol{\varepsilon}_f\} + \{\delta\boldsymbol{\varepsilon}_s\}^T[\mathbb{D}_s]\{\delta\boldsymbol{\varepsilon}_s\}\right)dzdydx \end{aligned} \tag{15}$$

where

$$\begin{aligned} [\mathbb{D}_f] &= \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \\ [\mathbb{D}_s] &= \frac{E}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & (1-\nu)/2 \end{bmatrix} \end{aligned} \tag{16}$$

The terms in equation (15) can be further studied by substituting equation (7) into equation (15), which yields

$$\int_x \int_y \left(\{\delta \hat{\boldsymbol{\varepsilon}}_f\}^T \left(\int_{-t/2}^{t/2} z^2 [\mathbb{D}_f] dz \right) \{\delta \hat{\boldsymbol{\varepsilon}}_f\} + \{\delta \boldsymbol{\varepsilon}_s\}^T \left(\int_{-t/2}^{t/2} [\mathbb{D}_s] dz \right) \{\delta \boldsymbol{\varepsilon}_s\} \right) dydx \quad (17)$$

By drawing an analogy between equations (17) and (9), it can be observed that

$$\{\delta \boldsymbol{M}\} = [\mathbb{D}_M]\{\delta \hat{\boldsymbol{\varepsilon}}_f\} \quad (18)$$

$$\{\delta \boldsymbol{Q}\} = [\mathbb{D}_Q]\{\delta \boldsymbol{\varepsilon}_s\} \quad (19)$$

where

$$\begin{gathered} [\mathbb{D}_M] = \int_{-t/2}^{t/2} z^2 [\mathbb{D}_f] dz = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \\ [\mathbb{D}_Q] = \int_{-t/2}^{t/2} [\mathbb{D}_s] dz = \frac{Etk_s}{1-\nu^2} \begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & (1-\nu)/2 \end{bmatrix} \end{gathered} \quad (20)$$

Analytically, k_s equals one. However, RS3 uses a value of 5/6, which is commonly adopted in the literature (e.g., Batoz et al., 1980).

5. Mindlin Plate: Weak form Linearization

Using the three constitutive equations (13,17,18) of the rate-form stress-strain relationships of the Mindlin plates, linearization of the Mindlin plate's weak form equation (9) is as follows

$$\psi_i(\{\boldsymbol{u}\}, \{\delta \boldsymbol{u}\}) + \delta\psi = 0 \quad (21)$$

where

$$\delta\psi = \int_x \int_y \left(\{\delta \hat{\boldsymbol{\varepsilon}}_f\}^T [\mathbb{D}_M] \{\delta \hat{\boldsymbol{\varepsilon}}_f\} + \{\delta \boldsymbol{\varepsilon}_s\}^T [\mathbb{D}_Q] \{\delta \boldsymbol{\varepsilon}_s\} + t\{\delta \boldsymbol{\varepsilon}\}^T [\mathbb{D}_m] \{\delta \boldsymbol{\varepsilon}\} \right) dydx \quad (22)$$

6. References

1. Batoz, J-L., Bathe, K-J. and Ho, L-W., A study of three-node triangular plate bending elements, International Journal for Numerical Methods in Engineering 15.12, 1771-1812, 1980.
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3. Owen, D. R. J. and Hinton, E., *Finite Elements in Plasticity: Theory and Practice*, Pineridge Press Limited, 1980.