# 1 Cylindrical Hole in an Infinite Elastic Medium

## 1.1 Problem Description

This problem verifies stresses and displacements for the case of a cylindrical hole in an infinite elastic medium subjected to a constant in-situ (compression +) stress field of:

$P_0$ = 30 MPa

The material is isotropic and elastic with the following properties:

Young’s modulus = 10,000 MPa
Poisson’s ratio = 0.2

The model was built in ***RS3***. The results can be directly compared to the 2D analytical solution as well as a ***Phase***$^{2}$ model. The radius of the hole is 1 m and is assumed to be small compared to the length of the cylinder. Therefore 2D plane strain conditions are in effect.

## 1.2 Closed Form Solution

The classical Kirsch solution can be used to find the radial and tangential displacement fields and stress distributions, for a cylindrical hole in an infinite isotropic elastic medium under plane strain conditions (e.g. see Jaeger and Cook, 1976 [1]).

The stresses $\sigma_r$, $\sigma_\theta$ and $\tau_{r\theta}$ for a point at polar coordinate ($r$,θ) near the cylindrical opening of radius $a$ (Figure 1-1) are given by:

$$\sigma_{rr} = P_0\left(1 - \frac{a^3}{r^3}\right)$$

$$\sigma_\theta = \frac{p_1 + p_2}{2}(1 + \frac{a^2}{r^2}) - \frac{p_1 - p_2}{2}(1 + \frac{3a^4}{r^4})\cos 2\theta$$

$$\tau_{r\theta} = -\frac{p_1 - p_2}{2}(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4})\sin 2\theta$$

The radial (outward) and tangential displacements (see Figure 1-1), assuming conditions of plane strain, are given by:

$$u_r = \frac{p_1 + p_2}{4G}\frac{a^2}{r} + \frac{p_1 - p_2}{4G}\frac{a^2}{r}[4(1-\nu) - \frac{a^2}{r^2}]\cos 2\theta$$

$$u_\theta = -\frac{p_1 - p_2}{4G}\frac{a^2}{r}[2(1-2\nu) + \frac{a^2}{r^2}]\sin 2\theta$$

where $G$ is the shear modulus and $\nu$ is Poisson’s ratio.

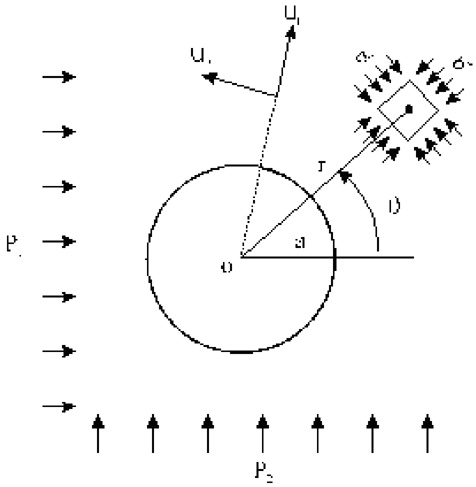

**Figure 1-1: Cylindrical hole in an infinite elastic medium**

## 1.3 Model Information

The model for this problem is built in ***RS3*** with a graded mesh of 10-noded tetrahedron elements as shown in Figure 1-2. The model uses the following parameters.

- excavation radius *a* = 1 m
- 40 segments (discretizations) around the circular opening
- extrusion length = 1 m
- fixed external boundary, located 21 m from the center
- z-restrained boundary conditions on front and back face

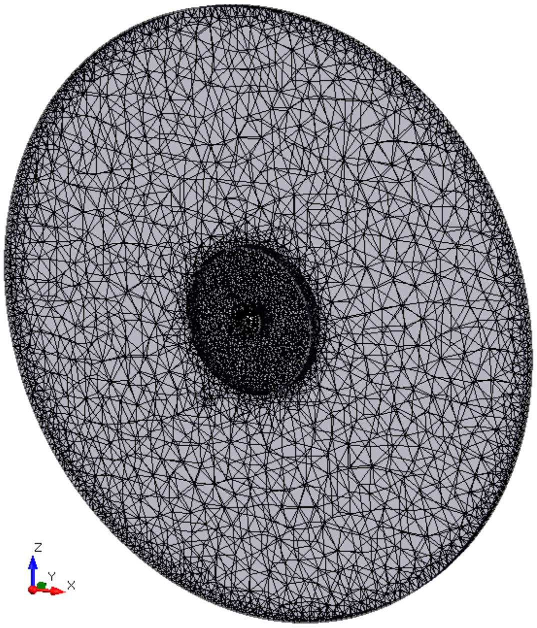

**Figure 1-2: Model of cylindrical hole in an infinite elastic medium in *RS3***

## 1.4 Results and Discussions

Figure 1-3, Figure 1-4 and Figure 1-5 show the radial stress, tangential stress, and radial displacement along a line (either the x- or y-axis) through the center of the models. The ***RS3*** results are in very close agreement with the analytical solutions and ***Phase²*** results.

Contour plots of the stresses and displacement are rendered in ***RS3***. The tangential stress ($\sigma_1$), radial stress ($\sigma_3$), and radial displacement distribution are presented in Figure 1-6, Figure 1-7, and Figure 1-8 respectively.

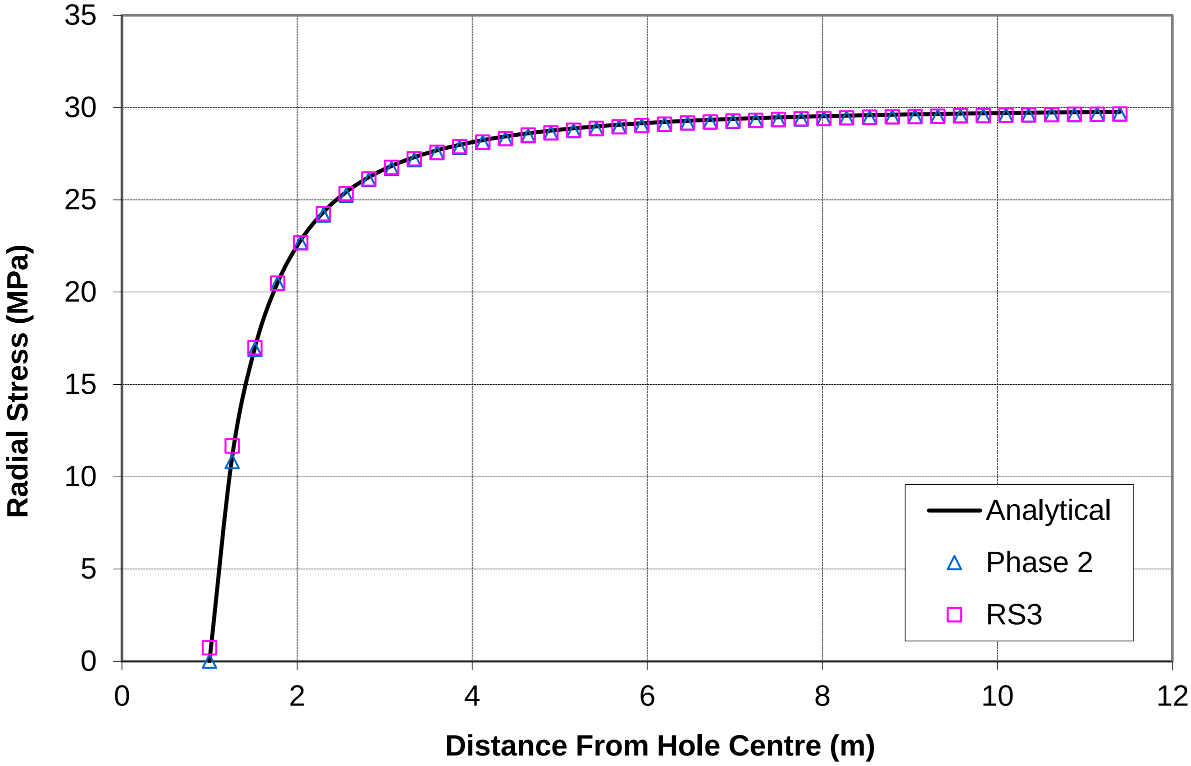

**Figure 1-3: Comparison of radial stresses for all solutions**

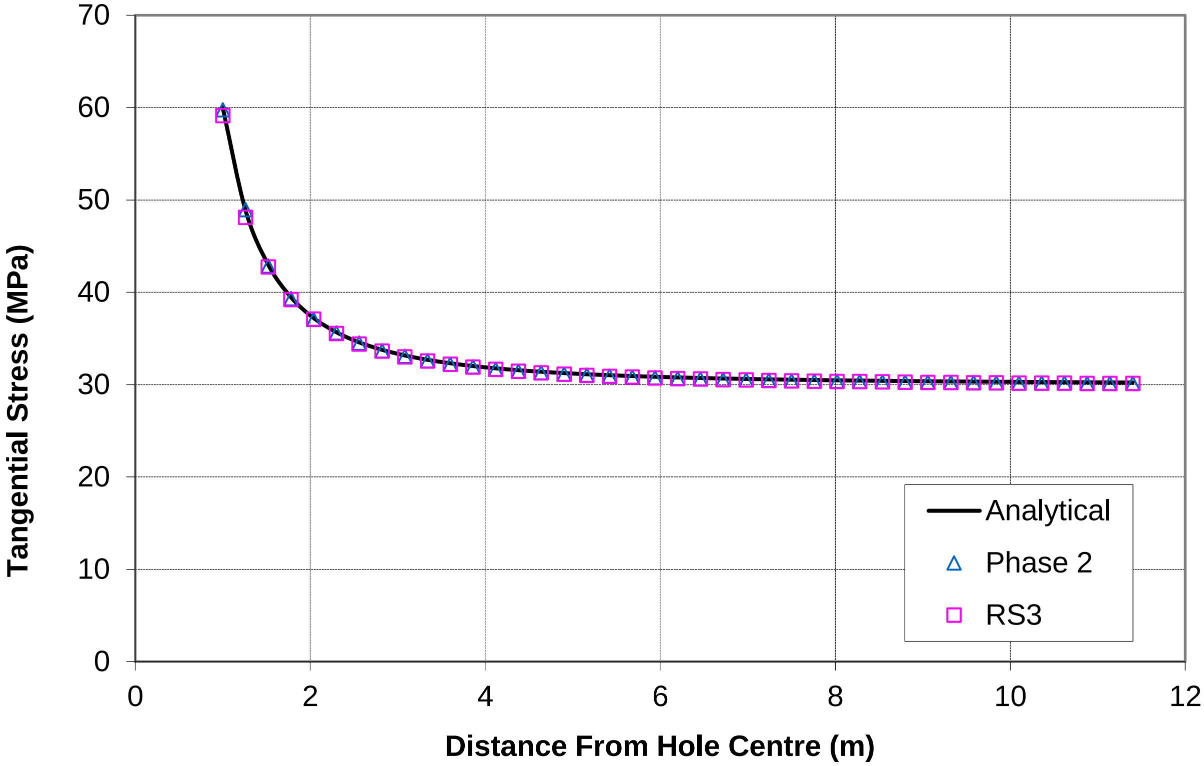

**Figure 1-4: Comparison of tangential stresses for all solutions**

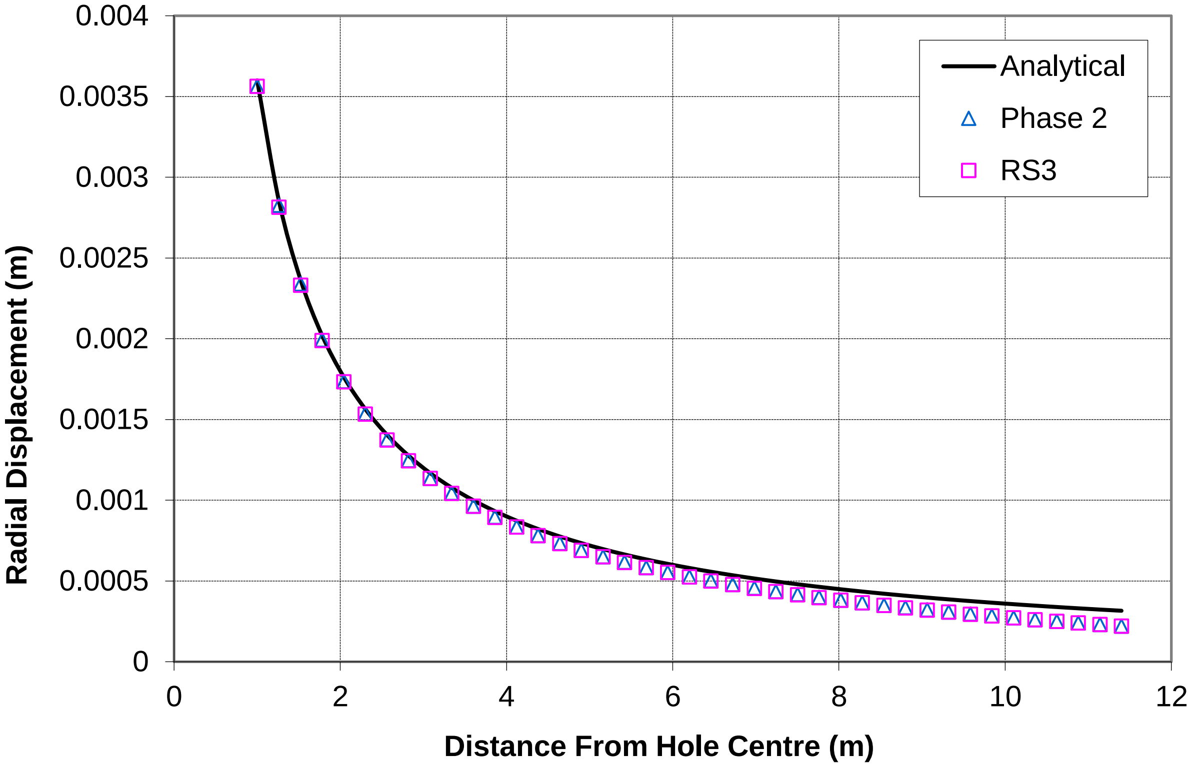

**Figure 1-5: Comparison of $u_r$ for all solutions**

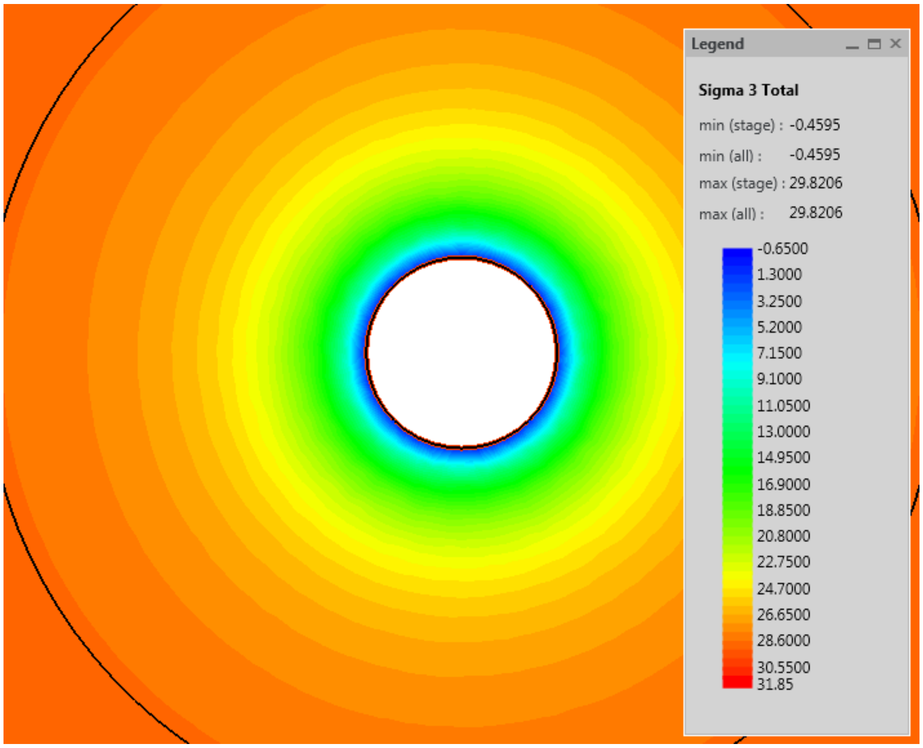

**Figure 1-6: Radial stress distribution for *RS3* results**

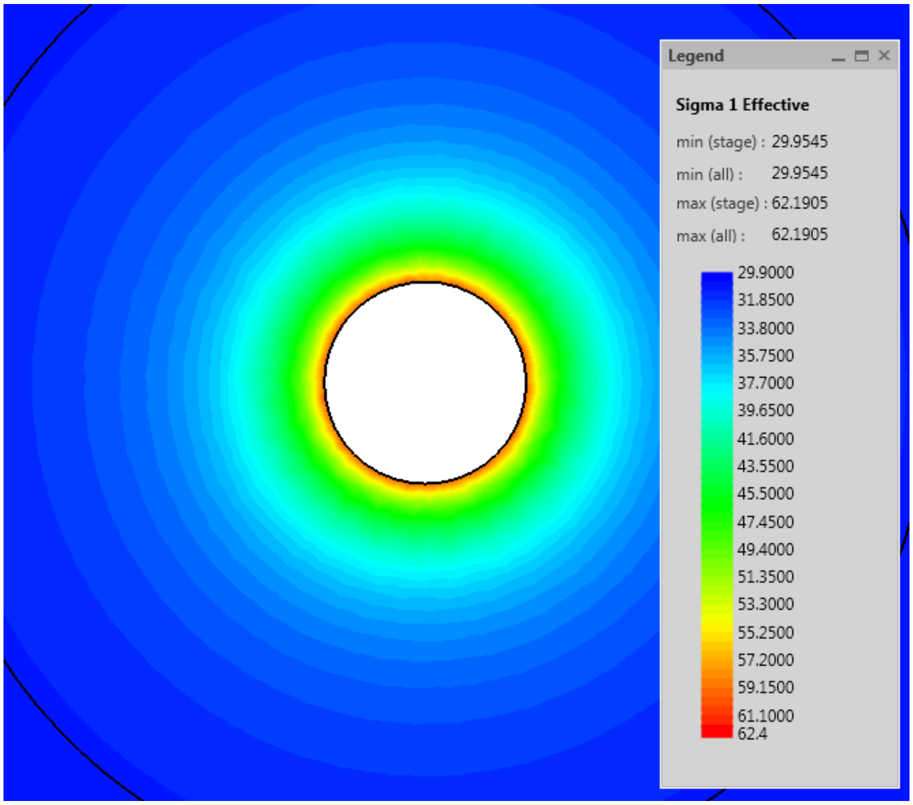

**Figure 1-7: Tangential stress distribution for *RS3* results**

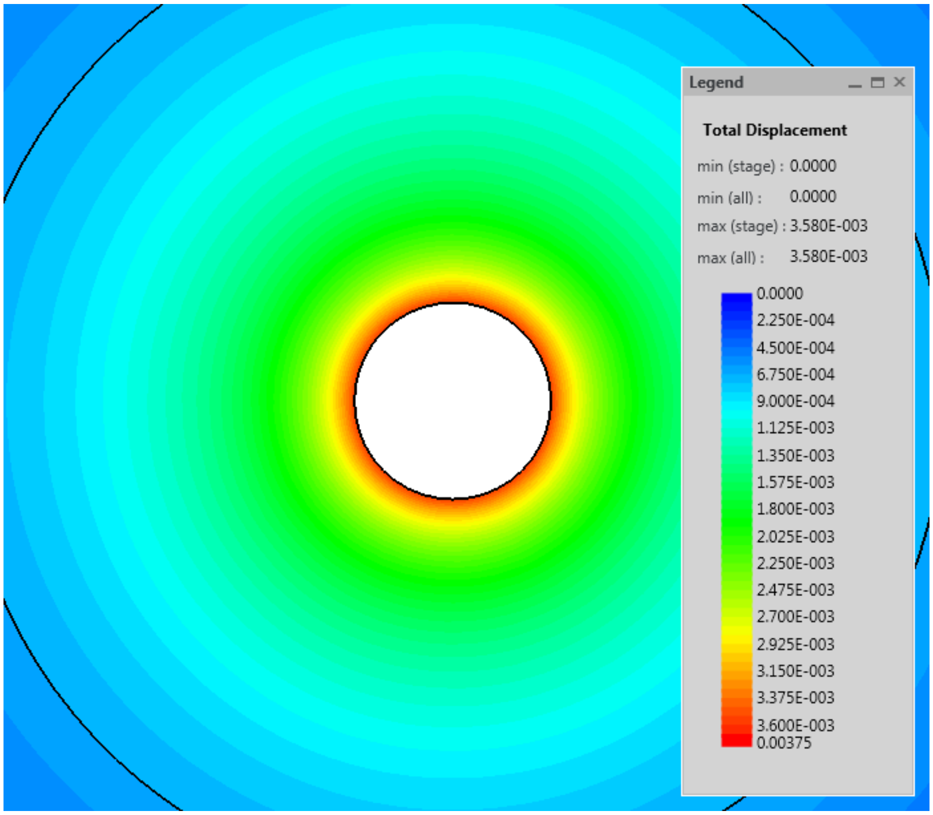

**Figure 1-8: Total displacement distribution for *RS3* results**

## 1.5 References

1. Jaeger, J.C. and N.G.W. Cook. (1976) *Fundamentals of Rock Mechanics,* 3rd Ed. London, Chapman and Hall.
2. O.C. Zienkiewicz. *The Finite Element Method in Engineering Science*, New York: McGraw Hill, 1971.

## 1.6 Data Files

The input data file **V001.rs3model** can be found in the ***RS3*** installation folder.