16 Groundwater Transient Flow

16.1 Problem Description

Case 1: 1-D Transient Saturated Flow

A totally saturated column of soil of height, L, and width, A, is initially at hydrostatic conditions of $H_{(t=0)} = H_0 = L$ until a total head of $H_{(z=L)} = H_T$ is applied at the top ($z = L$) and remains at this level for an indefinite period of time. The boundary condition at the bottom ($z = 0$) remains at the total head of $H_{(z=0)} = H_0 = L$.

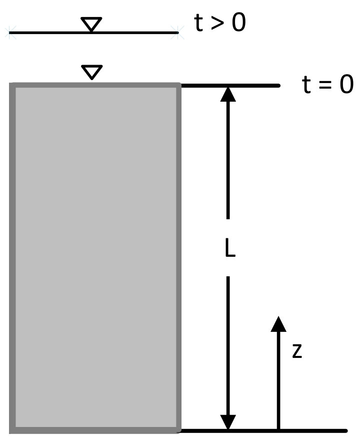

Figure 16-1: Total head of a column of soil for 1-D saturated flow analysis (case 1)

Case 2: 1-D Transient Unsaturated Flow

An initially dry column of soil of height, L, and width, A, is initially dry at constant pressure head $P_{(t=0)} = P_r$ until a pool of water from rainfall is applied at the top ($z = L$) such that the pressure head remains zero at the top for a period of time $P_{(z=L)} = 0$. The boundary condition at the bottom ($z = 0$) remains at the $P_{(z=0)} = P_r$.

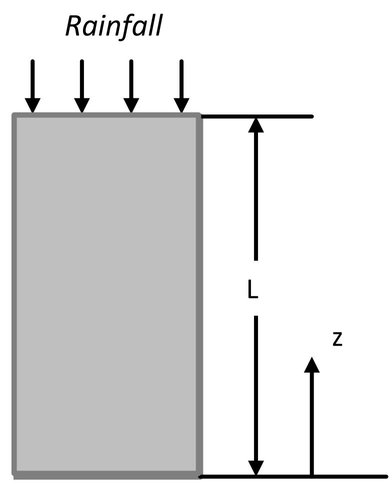

Figure 16-2: A column of soil for 1-D unsaturated flow analysis (case 2)

Case 3: 2-D Transient Unsaturated Flow

See Figure 16-3 for 2-D cross section of the soil column with dimension $A\times L$. The soil is initially dry at constant pressure head $P_{(t=0)} = P_r$. At t = 0, water is supplied such that a specified pressure head is applied at the top with pressure head set to zero in the middle and tapering rapidly to P_r at x = 0 and $x = A$ .The boundary condition at the bottom ($z = 0$) remains at the $P_{(z=0)} = P_r$. See Figure 16-4 for pressure head boundary condition at the top. Using symmetry about $x = A/2$, only half of the soil column is analyzed.

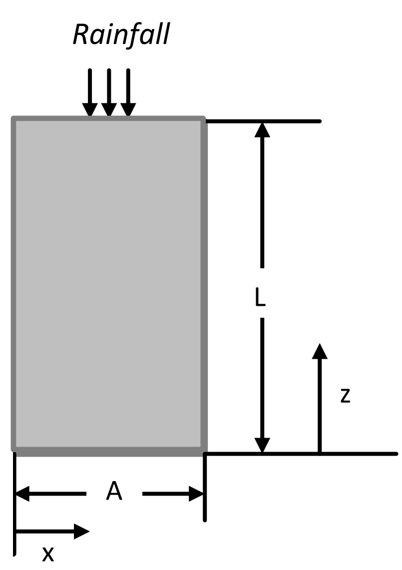

Figure 16-3: A column of soil for 2-D unsaturated flow analysis (case 3)

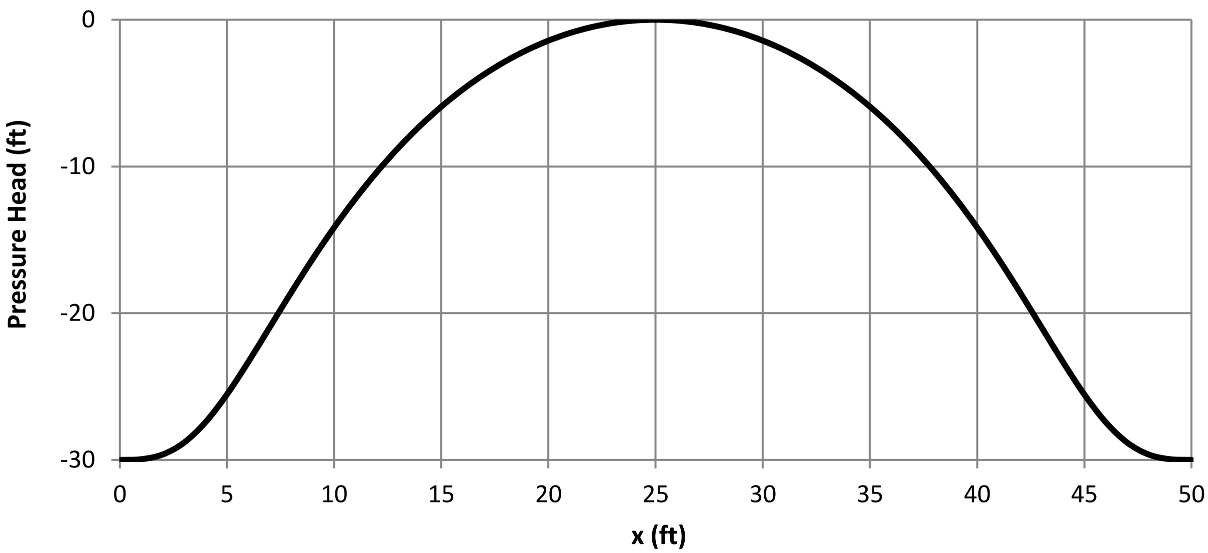

Figure 16-4: Pressure head boundary condition at the top of soil sample distributed across *x*

The model for each of the three cases is constructed separately in ***RS3*** with uniform mesh of 10-noded tetrahedron elements. Figure 16-5 shows case 1 as constructed in ***RS3*** with boundary conditions applied. Table 16-1 summarizes the model parameters for each of the three cases.

Table 16-1: Model parameters

Case 1 Parameters	*Case 2 Parameters*	*Case 3 Parameters*
$L = 50$ ft $H_0 = 50$ ft $H_T = 75$ ft $k_s = 0.1$ ft/day $m_v = 0.00001$ ft^2/lb $\gamma_w = 62.4$ lb/ft^3	$L = 50$ ft $P_r = -30$ ft $k_s = 0.1$ ft/day $\alpha = 0.1$ day^{-1} $\theta_r = 0.15$ $\theta_s = 0.45$ $\gamma_w = 62.4$ lb/ft^3	$A = L = 50$ ft $P_r = -30$ ft $k_s = 0.1$ ft/day $\alpha = 0.1$ day^{-1} $\theta_r = 0.15$ $\theta_s = 0.45$ $\gamma_w = 62.4$ lb/ft^3

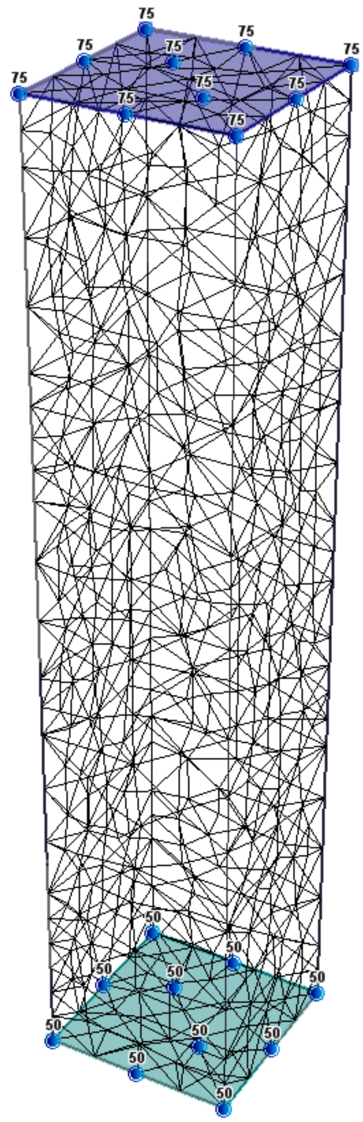

Figure 16-5: Soil sample constructed in *RS3* (Case 1)

16.2 Analytical Solution

Richards' equation can be used to determine the total head as a function of location and time, $H_{(z,t)}$. The analytical solution for the box-shaped soil sample is provided by Fred T. Tracy (2011). The solutions for $H_{(z,t)}$ are presented below for each condition:

Case 1: 1-D Transient Saturated Flow

$$H(z,0) = H_0 = L$$

$$H(L,t) = H_T$$

$$H(0,t) = H_0 = L$$

$$c = \frac{m_v \gamma_w}{k_s}$$

$$\lambda_k = \frac{\pi k}{L} \;,\; k = 0,\, 1,\, 2,\, \ldots$$

$$\mu_k = \frac{\lambda_k^2}{c}$$

$$H_{(z,t)} = H_0 + \frac{H_T - H_0}{L}\left[z + \sum_{k=1}^{\infty} (-1)^k \left(\frac{2}{\lambda_k} \right) \sin(\lambda_k z) e^{-\mu_k t} \right]$$

Parameters for Unsaturated Flow Analysis:

Hydraulic conductivity

$k = k_s k_r$

k_s = saturated hydraulic conductivity

k_r = relative hydraulic conductivity

Gardner's equation

$k_r = e^{\alpha P}$

α = positive parameter

P = matric suction head

Moisture content

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = k_r$$

θ_r = residual moisture content

θ_s = saturated moisture content

S_e = effective saturation

$$c = \frac{\alpha(\theta_s - \theta_r)}{k_s}$$

$$\varepsilon = e^{\alpha P_r}$$

Case 2: 1-D Transient Unsaturated Flow

$$P(z,0) = P_r$$
$$P(L,t) = 0$$
$$P(0,t) = P_r$$
$$\lambda_k = \frac{\pi k}{L}\ ,\ \ k = 0,\ 1,\ 2,\ \ldots$$

$$\mu_k = \frac{1}{c}\left(\frac{\alpha^2}{4} + \lambda_k^2\right)$$

$$\bar{P}_{(z,t)} = (1-\varepsilon)e^{\frac{\alpha}{2}(L-z)}\left[\frac{\sinh\left(\frac{\alpha}{2}z\right)}{\sinh\left(\frac{\alpha}{2}L\right)} + \frac{2}{Lc}\sum_{k=1}^{\infty}(-1)^k\left(\frac{\lambda_k}{\mu_k}\right)\sin(\lambda_k z)e^{-\mu_k t}\right]$$

$$H_{(z,t)} = \frac{1}{\alpha}\ln\left(\bar{P}+\varepsilon\right) + z$$

Case 3: 2-D Transient Unsaturated Flow

$$P(x,z,0) = P_r$$

$$P(0,z,t) = P(A,z,t) = P(x,0,t) = P_r$$

$$P(x,L,t) = \frac{1}{\alpha}\ln\left\{\varepsilon + (1-\varepsilon)\left[\frac{3}{4}\sin\left(\frac{\pi}{A}x\right) - \frac{1}{4}\sin\left(\frac{3\pi}{A}x\right)\right]\right\}$$

$$\lambda_k = \frac{\pi k}{L}\ ,\ \ k = 0,\ 1,\ 2,\ \ldots \qquad \lambda_i = \frac{\pi i}{A}\ ,\ \ i = 0,\ 1,\ 2,\ \ldots \qquad \beta_i = \sqrt{\frac{\alpha^2}{4} + \lambda_i^2} \qquad \gamma_{ik} = \frac{1}{c}\left(\beta_i^2 + \lambda_k^2\right)$$

$$\bar{P}(x,z,t) = (1-\varepsilon)e^{\frac{\alpha}{2}(L-z)}\left[\begin{array}{l}\frac{3}{4}\sin\left(\frac{\pi}{A}x\right)\left[\frac{\sinh(\beta_1 z)}{\sinh(\beta_1 L)} + \frac{2}{Lc}\sum_{k=1}^{\infty}(-1)^k\left(\frac{\lambda_k}{\gamma_{1k}}\right)\sin(\lambda_k z)e^{-\gamma_{1k}t}\right] - \\ \frac{1}{4}\sin\left(\frac{3\pi}{A}x\right)\left[\frac{\sinh(\beta_3 z)}{\sinh(\beta_3 L)} + \frac{2}{Lc}\sum_{k=1}^{\infty}(-1)^k\left(\frac{\lambda_k}{\gamma_{3k}}\right)\sin(\lambda_k z)e^{-\gamma_{3k}t}\right]\end{array}\right]$$

$$H_{(z,t)} = \frac{1}{\alpha}\ln\left(\bar{P}+\varepsilon\right) + z$$

16.3 Results

Figure 16-6 to Figure 16-8 show the total head along z-axis. For case 3, unsaturated flow 2-D, the query line along z-axis goes through *x* = 25 ft. The ***RS3*** results are in very close agreement with the analytical solutions.

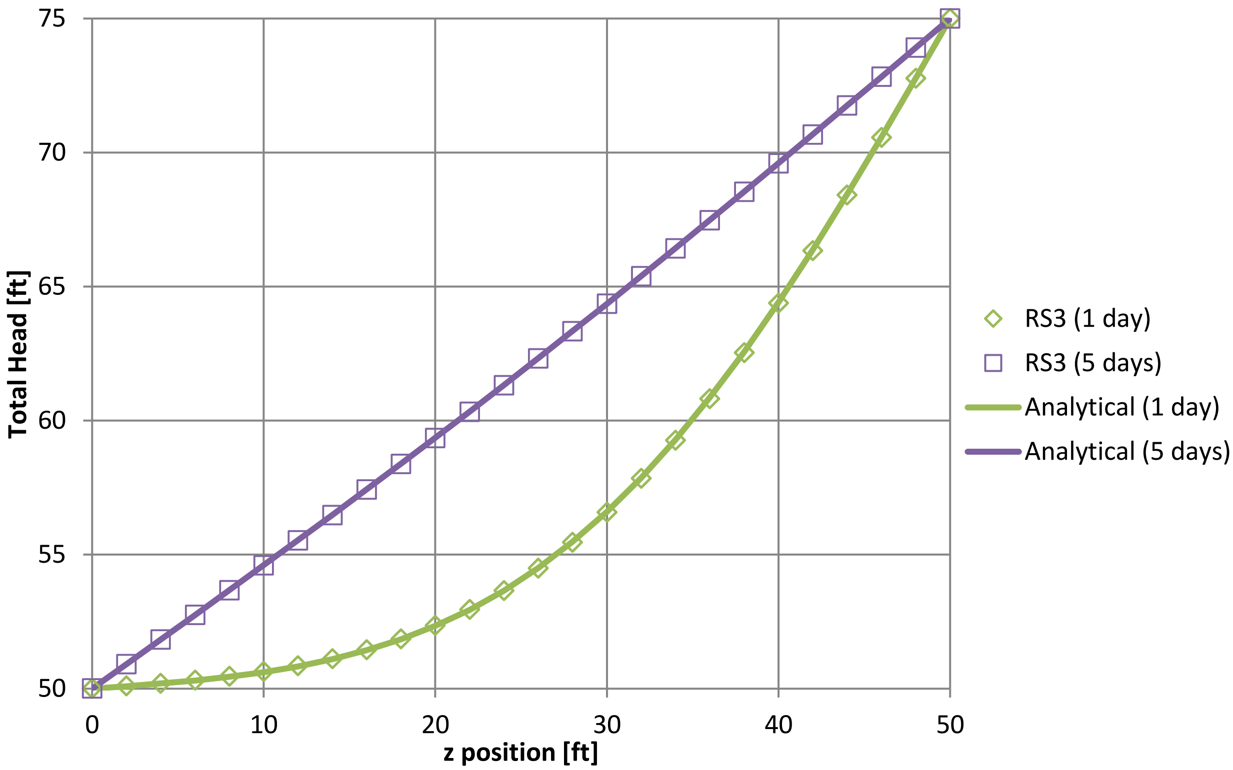

Figure 16-6: Total head along height of soil column for saturated flow 1-D

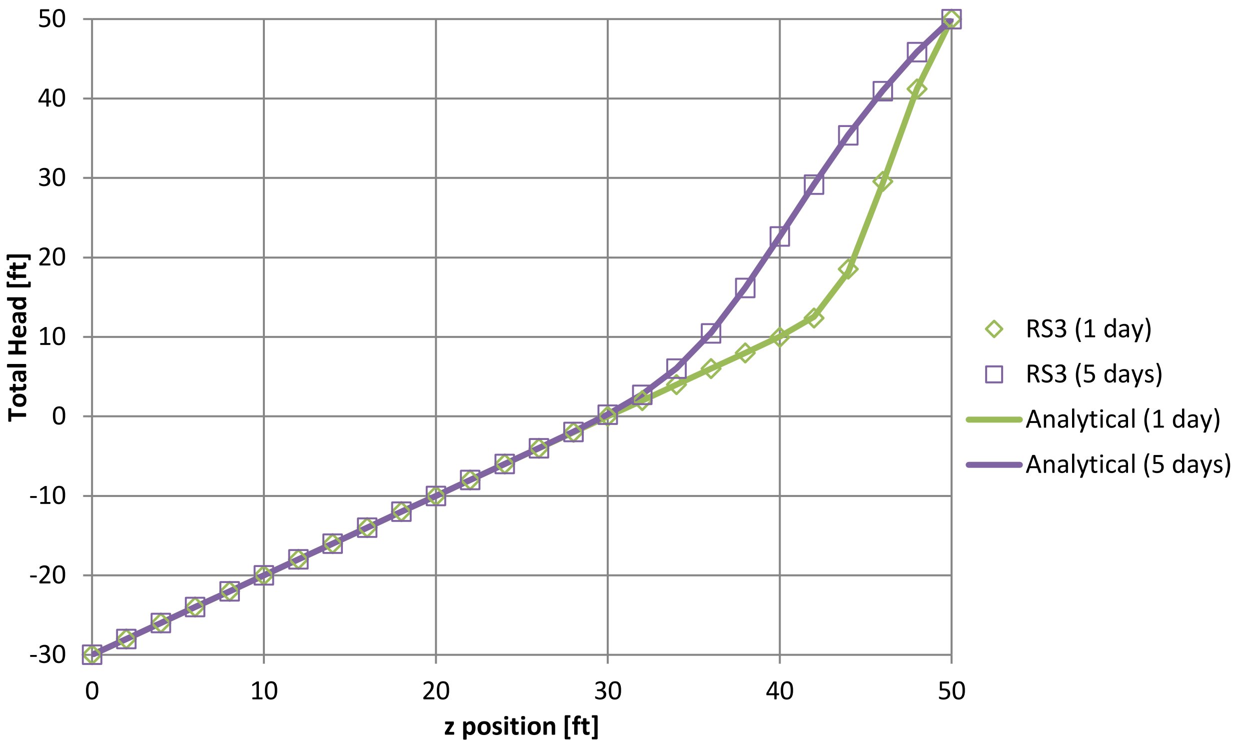

Figure 16-7: Total head along height of soil column for unsaturated flow 1-D

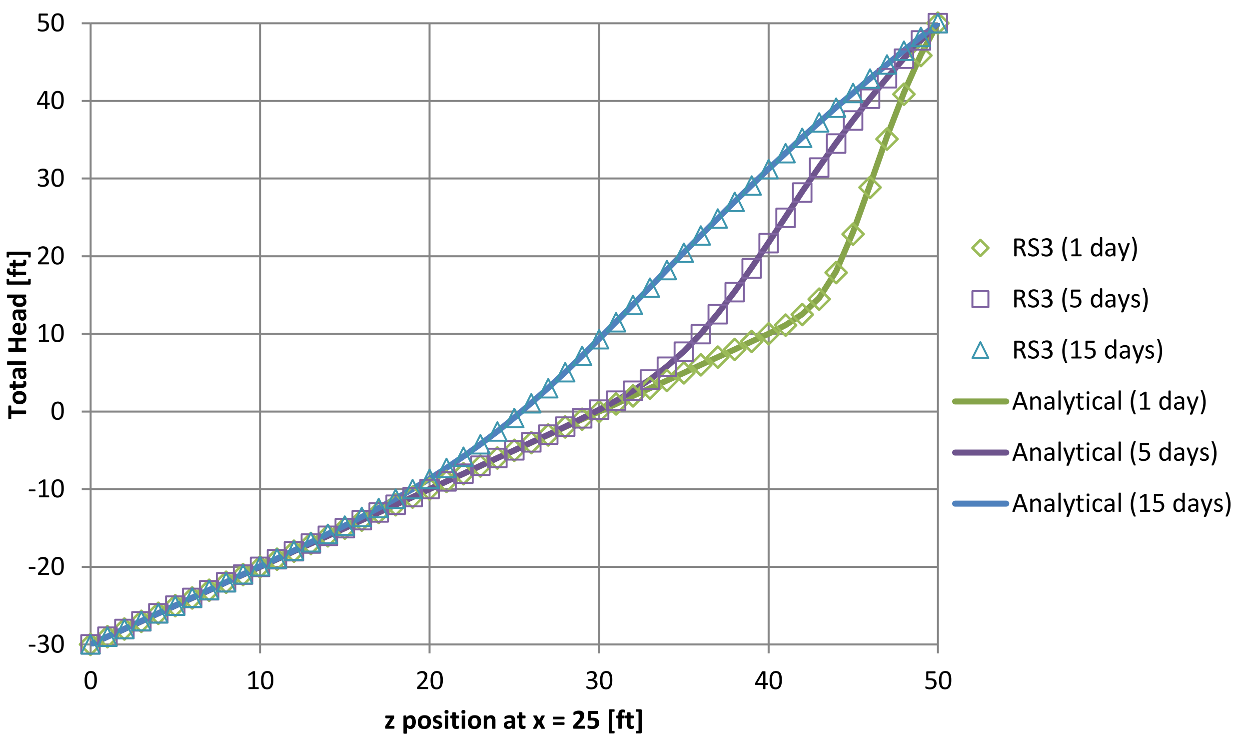

Figure 16-8: Total head along height of soil layer for unsaturated flow 2-D, at *x* = 25

16.4 References

1. Fred T. Tracy (2011). Analytical and Numerical Solutions of Richards' Equation with Discussions on Relative Hydraulic Conductivity. *Hydraulic Conductivity - Issues, Determination and Applications*. Prof. Lakshmanan Elango (Ed.). InTech.

2. Tracy, F. (2007). Three-dimensional analytical solutions of Richards' equation for a boxshaped soil sample with piecewise-constant head boundary conditions on the top. *Journal of Hydrology*, Vol. 336, pp. 391-400.

16.5 Data Files

The input data files can be found in the ***RS3*** installation folder:

Table 16-2: Input data files for transient groundwater flow

Case	*File name*
1	**V016 Saturated 1D.rs3model**
2	**V016 Unsaturated 1D.rs3model**
3	**V016 Unsaturated 2D.rs3model**