ROCSCIENCE

RocFall3

Rigid Body Basic Physics

(Legacy Formulation for Version 1.005 and Earlier)

Theory Manual

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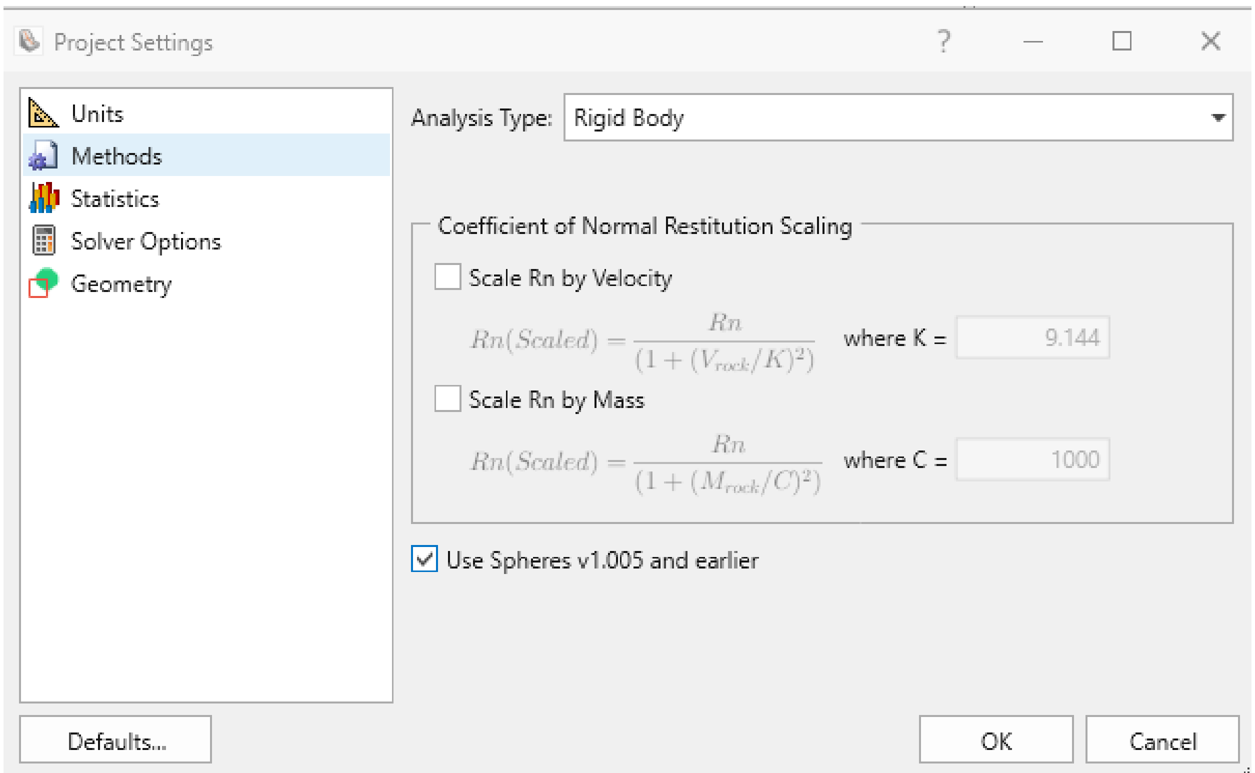
Introduction

This theory manual documents the basic calculations used in **RocFall3 version 1.005 and earlier** to simulate the rock paths as rocks travel down the slope. RocFall3 has 2 engine modes: lumped mass and rigid body. The lumped mass mode assumes the rock as a particle (point) and the rigid body mode considers the rock's actual shape and the effects from the shape's size and angular momentum.

This theory manual documents calculations for the legacy Rigid Body method used in version 1.005 and earlier. If using a newer version of RocFall3, this legacy formulation will be used only if the analysis type is Rigid Body and "Use Spheres v1.005 and earlier" is selected.



Rockfall trajectories consist of 3 main parts: projectile, impacts and sliding/rolling.

1. Projectile

The physics for projectile motion of the rock is the same for both the lumped mass and rigid body modes. While a rock is flying through the air, there are no forces acting on the rock except for gravity. For the rock at any given time $\boldsymbol{t}$, let's define the following variables:

$\vec{\boldsymbol{p}}$ **=** position (x,y,z) vector

$\overleftrightarrow{\boldsymbol{R}}_{\boldsymbol{r}}$= rotation matrix

$\vec{\boldsymbol{v}}$ = velocity (x,y,z) vector

$\vec{\boldsymbol{\omega}}$ = angular velocity (x,y,z) vector

With only gravity as the force, $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_{\boldsymbol{g}} = \boldsymbol{M} \cdot \vec{\boldsymbol{g}} = \boldsymbol{M} \cdot (0, 0, -9.81)$.

We can then write:

$$\vec{\boldsymbol{p}}(\boldsymbol{t}) = \vec{\boldsymbol{v}}_{\boldsymbol{o}} \cdot \boldsymbol{t} + \frac{1}{2}\vec{\boldsymbol{g}} \cdot \boldsymbol{t}^{2} \qquad \text{......(1.1)}$$

$$\vec{\boldsymbol{v}}(\boldsymbol{t}) = \vec{\boldsymbol{v}}_{\boldsymbol{o}} + \vec{\boldsymbol{g}} \cdot \boldsymbol{t} \qquad \text{......(1.2)}$$

And with $\vec{\boldsymbol{\omega}}$ and $\boldsymbol{t}$, we can get the rotating axis $\vec{\boldsymbol{u}} = \frac{\vec{\boldsymbol{\omega}} \cdot \boldsymbol{t}}{|\vec{\boldsymbol{\omega}} \cdot \boldsymbol{t}|}$ and the angle to be rotated is $\theta = ||\vec{\boldsymbol{\omega}} \cdot \boldsymbol{t}||$. We can then construct the rotation matrix [1]:

$$R = \begin{bmatrix} cos\theta + u_x^2(1 - cos\theta) & u_x u_y(1 - cos\theta) - u_z sin\theta & u_x u_z(1 - cos\theta) + u_y sin\theta \\ u_y u_x(1 - cos\theta) + u_z sin\theta & cos\theta + u_y^2(1 - cos\theta) & u_y u_z(1 - cos\theta) - u_x sin\theta \\ u_z u_x(1 - cos\theta) - u_y sin\theta & u_z u_y(1 - cos\theta) + u_x sin\theta & cos\theta + u_z^2(1 - cos\theta) \end{bmatrix}$$

Updated rotation of the rock after time $\boldsymbol{t}$ is then:

$$\overleftrightarrow{\boldsymbol{R}}_{\boldsymbol{r}}(\boldsymbol{t}) = \boldsymbol{R} \cdot \overleftrightarrow{\boldsymbol{R}}_{\boldsymbol{ro}} \cdot \boldsymbol{R}^{\boldsymbol{T}} \qquad \text{......(1.3)}$$

2. Impacts

The Rigid Body method in **RocFall3** assumes both the rock and the slope are rigid bodies during impacts. It is assumed that there are no deformations nor break-ups. The impact theory is based on non-smooth dynamics developed by Dr. JJ. Moreau [2]. The impact is viewed as a non-smooth event where the velocities are allowed to change or jump instantaneously. The change of velocity over time is not a continuous/smooth function at the time of impact.

Before the impact calculations, we rotate the rocks to a 1/2/3 (first tangential, second tangential and normal directions) frame.

Impulses (dP) or the change of momentum (mass x velocity) are key to determine the outgoing velocities of the rock after impacts. We can write the relationship [4] as:

$$d\vec{v} = m^{-1}d\vec{p} \qquad \text{……………………………(2.3a)}$$

where $d\vec{v}$ is the change in relative velocity (velocities between the slope and the rock), m is the effective mass and $d\vec{p}$ is the relative impulse. Since the slope doesn't move and that its mass approaches infinite comparing to the rock, in figuring out rockfall impact problems, $d\vec{v}$ is just the change in velocity of the rock and the effective mass m is just the mass (M) of the rock.

$$d\vec{V} = M^{-1}d\vec{P} \qquad \text{……………………………(2.3b)}$$

The above velocities ($d\vec{V}$) are at the centre of mass of the rock. During impact, change in angular momentum ($d\vec{L}$) can affect the rock's rotations as well.

$$d\vec{\omega} = \overleftrightarrow{I}^{-1}d\vec{L} \qquad \text{……………………………(2.4)}$$

where $\vec{\omega}$ is the angular velocities and $\overleftrightarrow{I}$ is the moment of inertia matrix of the rock. Change in angular momentum can be related to the impulses by the moment arm (r):

$$d\vec{L} = \overleftrightarrow{r} \cdot d\vec{P} \qquad \text{……………………………(2.5)}$$

where $\overleftrightarrow{r}$ is the rotational matrix from the centre of mass to the contact point.

$$r = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \qquad \text{……………………………(2.6)}$$

where $\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = COM - contact\ point$

combine (2.4) and (2.5) we get:

$$d\vec{\omega} = \overleftrightarrow{I}^{-1} \cdot \overleftrightarrow{r}^{T} \cdot d\vec{P} \qquad \text{……………………………(2.7)}$$

The relationship between the rock's contact point velocity ($\vec{\gamma}$) and the centre of mass velocity ($\vec{V}$) and be expressed as follows:

$$\vec{\gamma} = \vec{V} + \vec{r} \cdot \vec{\omega} \qquad \text{……………………………(2.8)}$$

To further simplify, we first define:

$$w = \begin{bmatrix} 1 & 0 & 0 & 0 & -r_3 & r_2 \\ 0 & 1 & 0 & r_3 & 0 & -r_1 \\ 0 & 0 & 1 & -r_2 & r_1 & 0 \end{bmatrix}$$

w is simply a rotational matrix that transforms centre of mass velocities into contact point velocities:

$$\vec{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \overleftrightarrow{w} \cdot \vec{V} \qquad \text{.................................(2.9)}$$

Then we define a mass and inertia matrix:

$$M = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{11} & I_{21} & I_{31} \\ 0 & 0 & 0 & I_{12} & I_{22} & I_{32} \\ 0 & 0 & 0 & I_{13} & I_{23} & I_{33} \end{bmatrix}$$

We make $\overleftrightarrow{G} = w \cdot M^{-1} \cdot w^T$ and we can relate contact point velocities directly with impulses

$$d\vec{\gamma} = \overleftrightarrow{G} \cdot d\vec{P} \qquad \text{.................................(2.10)}$$

If we know the change in contact velocities, we can obtain $d\vec{P}$. With $d\vec{P}$, from (2.3b) and (2.7) we can find the centre of mass translational and angular velocities after impact.

There are many theories are how to obtain change in contact point velocities ($d\vec{\gamma}$). One of the most popular theory is through the use of coefficient of restitution ε.

$$\varepsilon = \begin{bmatrix} r_T & 0 & 0 \\ 0 & r_T & 0 \\ 0 & 0 & r_N \end{bmatrix}$$

Newton's experimental law of Impacts states:

$$\vec{\gamma}' = \varepsilon \, \vec{\gamma} \qquad \text{.................................(2.11)}$$

where γ is the incoming velocity and γ' is the outgoing velocity.

$$d\vec{\gamma} = \epsilon \cdot \vec{\gamma} - \vec{\gamma} = (\varepsilon - 1)\vec{\gamma} \qquad \text{.................................(2.12)}$$

To summarize, to calculate rigid body impacts, we first obtain the contact point velocities from (2.8). Solve for change in contact point velocities with (2.12). Plug it into (2.10) to get $d\vec{P}$. With $d\vec{P}$ use (2.3b) and (2.7) to get $d\vec{V}$ and $d\vec{\omega}$.

One thing to note that Newton's experimental law only described the impact in the normal direction. The impulse in the tangential direction is limited by the frictional force, as described by Coulomb's Law, in the direction opposite to incoming tangential velocities. Therefore,

$$dP_t = \mu \cdot dP_n \qquad \text{.................................(2.13)}$$

However, it's been observed this constraint limits the amount of energy loss in the tangential direction and is often insufficient to correctly describe rocks' impacts with slopes. Afterall, rocks and slopes are hardly infinitely rigid. Moreau [2] introduced the tangential coefficient of restitutions (ε_T) with "inelastic or …soft" impacts with Nonsmooth Dynamics. We decided to make this a project settings option, so the user can decide whether to apply this cap on tangential impulses.

3. Sliding/Rolling

The physics for rigid body rolling and sliding are the same as that of the lumped mass mode, except that the friction acts on the contact point and the gravitational force acts on the centre of mass. The eccentric frictional force affects not just the rock's translational velocity but also its angular velocity. The diagram below [3] illustrates the forces acting on a rolling 2D body:

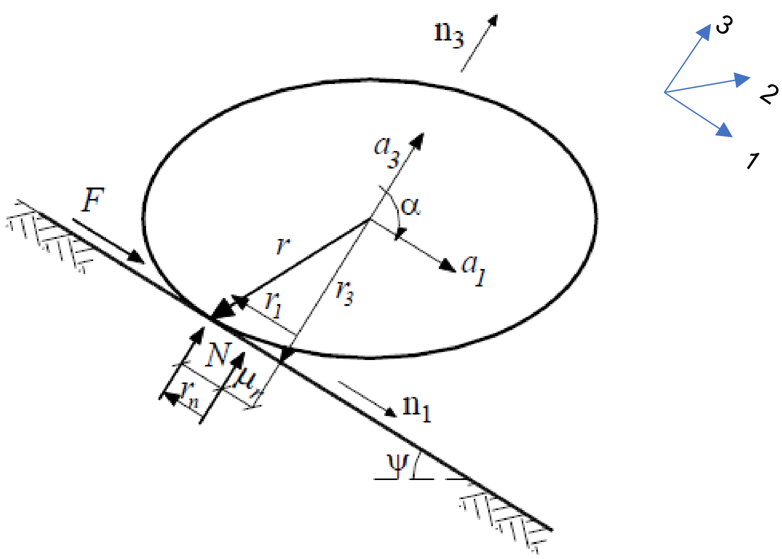


Figure 3.1: Rigid body 2D object rolling (Figure 4.19 Ashayer [3])

3.1.1. Rolling

For a rock to be considered pure rolling, the tangential contact point velocities must be very small or close to 0. Contact point velocities can be calculated per equation 2.8. $\boldsymbol{F}$ in Figure 3.1 above is the unknown frictional force that includes rolling friction and static friction. For rolling condition to hold, we need to compare $\boldsymbol{F}$ to the dynamic friction force to make sure it's smaller after all the forces are solved.

$$|F| \leq \mu \cdot N \qquad \text{(3.9)}$$

If equation 3.9 is not met, the rock is considered to start sliding during rolling. (See next section.)

We first define the following:

$$\boldsymbol{g_1} = \vec{\boldsymbol{g}} \cdot \overrightarrow{\boldsymbol{t_1}}$$

$$\boldsymbol{g_2} = \vec{\boldsymbol{g}} \cdot \overrightarrow{\boldsymbol{t_2}}$$

$$\boldsymbol{g_3} = \vec{\boldsymbol{g}} \cdot \overrightarrow{\boldsymbol{N}}$$

where $\vec{\boldsymbol{g}} = (0, 0, -9.81)$

$$\vec{\boldsymbol{r}} = \boldsymbol{COM - contact\ point}$$

$\boldsymbol{a}$ *is the COM acceleration*

$\boldsymbol{\alpha}$ *is the COM angular acceleration*

We re-write equation 4.5.1 Ashayer [3] to include the extra dimension in the balancing equations. The equations follow the same t1/t2/n frame.

1) $M \times g1 + F1 = M \times a1$
2) $M \times g2 + F2 = M \times a2$
3) $N + M \times g3 = M \times a3$
4) $-F1(r3) + N(r1) = I22 \times \alpha2$
5) $F2(r3) + N(-r1) = I11 \times \alpha1$
6) $F1(r2) - F2(r1) = I33 \times \alpha3$
7) $a1 = r3 \times \alpha2 - r2 \times \alpha3$
8) $a2 = -r3 \times \alpha1 + r1 \times \alpha3$
9) $a3 = -r1 \times \alpha2 + r2 \times \alpha1$

……(3.10)

We have 9 equations and 9 unknowns ($a1, a2, a3\ \alpha1, \alpha2, \alpha3, F1, F2, and\ N$).

Equations 3.10 describe the case where there's no rolling friction. To introduce rolling friction we replace equations 1 and 2 in 3.10 with:

1) $M \times g1 + F1 + \widehat{n_1} \times p1 \times \mu_r \times N = M \times a1$
2) $M \times g2 + F2 + \widehat{n_2} \times p2 \times \mu_r \times N = M \times a2$

……(3.10b)

where

$$\widehat{n_2} = \omega_1/|\omega_1|$$

$$\widehat{n_1} = -\omega_2/|\omega_2|$$

$$p1 = \frac{|\gamma_1|}{\sqrt{\gamma_1^2+\gamma_2^2}} \quad \text{and}$$

$$p2 = \frac{|\gamma_2|}{\sqrt{\gamma_1^2+\gamma_2^2}}$$

……(3.11)

3.1.2. Rolling-Sliding

Rolling-Sliding mode occurs when tangential contact point velocities are not 0 or if the solved static friction force ($\boldsymbol{F}$) is greater than dynamic friction ($\mu \cdot N$). During sliding, the frictional force is known and is constant is opposite to the direction of travel [3].

$$F_{FR} = \mu \cdot N$$

We replace equations 7 and 8 in 3.10 with:

$$\left.\begin{aligned} F1 &= \frac{-\gamma_1}{|\gamma_1|} \times p1 \times \mu \times N + \widehat{n_1} \times p1 \times \mu_r \times N \\ F2 &= \frac{-\gamma_2}{|\gamma_2|} \times p2 \times \mu \times N + \widehat{n_2} \times p2 \times \mu_r \times N \end{aligned}\right\} \quad \ldots\ldots(3.10c)$$

where $\widehat{n_1}$, $\widehat{n_2}$, $p1$ and $p2$ can be calculated by equations 3.11.

Equation 3.10 are now down to 7 equations with 7 unknowns ($a1, a2, a3, \alpha1, \alpha2, \alpha3\ and\ N$).

4. References

[1] Taylor, Camillo J.; Kriegman, David J. (1994). *"Minimization on the Lie Group SO(3) and Related Manifolds"*. Technical Report No. 9405. Yale University.

[2] Moreau J.J. (1988) "Unilateral Contact and Dry Friction in Finite Freedom Dynamics". In: *Moreau J.J., Panagiotopoulos P.D. (eds) Nonsmooth Mechanics and Applications. International Centre for Mechanical Sciences (Courses and Lectures), vol 302*. Springer, Vienna.

[3] Ashayer, P. (2007). "Application of Rigid Body Impact Mechanics and Discrete Element Modeling to Rockfall Simulation". *PhD Thesis*, Department of Civil Engineering, University of Toronto, Ontario, Canada.